ESTIMATION OF $p$-ADIC SIZES OF COMMON ZEROS OF PARTIAL DERIVATIVE POLYNOMIALS ASSOCIATED WITH A QUINTIC FORM

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Abstract. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be a vector in a space $\mathbb{Z}^n$ with $\mathbb{Z}$ ring of integers and let $q$ be a positive integer, $f$ a polynomial in $\mathbf{x}$ with coefficients in $\mathbb{Z}$. The exponential sum associated with $f$ is defined as $S(f; q) = \sum \exp (2\pi i f(x)/q)$ where the sum is taken over a complete set of residues modulo $q$. The value of $S(f; q)$ has been shown to depend on the estimate of the cardinality $|V|$, the number of elements contained in the set $V = \{x \mod q \mid f_x \equiv 0 \mod q\}$ where $f_x$ is the partial derivatives of $f$ with respect to $x$. To determine the cardinality of $V$, the information on the $p$-adic sizes of common zeros of the partial derivatives polynomials need to be obtained. This paper discusses a method of determining the $p$-adic sizes of the components of $\{\xi, \eta\}$ a common root of partial derivative polynomials of $f(x, y)$ in $\mathbb{Z}_p[x, y]$ of degree five based on the $p$-adic Newton polyhedron technique associated with the polynomial. The quintic polynomial is of the form $f(x, y) = ax^5 + bx^4 y + cx^3 y^2 + dx^2 y^3 + exy^4 + my^5 + nx + ty + k$.

Keywords: Exponential sums, cardinality, $p$-adic sizes, Newton polyhedron

Abstrak. Katakkan $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ vektor dalam ruang $\mathbb{Z}^n$ dengan $\mathbb{Z}$ menandakan gelanggang integer dan $q$ integer positif, $f$ polinomial dalam $\mathbf{x}$ dengan pekali dalam $\mathbb{Z}$. Hasil tambah eksponen yang disekutukan dengan $f$ ditakrifkan sebagai $S(f; q) = \sum \exp (2\pi i f(x)/q)$ yang dinilai bagi semua nilai $x$ di dalam reja lengkap modulo $q$. Nilai $S(f; q)$ adalah bersandar kepada penganggaran bilangan unsur $|V|$, yang terdapat dalam set $V = \{x \mod q \mid f_x \equiv 0 \mod q\}$ dengan $f_x$ menandakan polinomial-polinomial terbitan separa $f$ terhadap $x$. Untuk menentukan kekardinalan bagi $V$, maklumat mengenai saiz $p$-adic pensifar sepunya perlu diperolehi. Makalah ini membincangkan suatu kaedah penentuan saiz $p$-adic pensifar sepunya $f(x, y)$ dalam $\mathbb{Z}_p[x, y]$ berdasarkan lima bersasakan teknik polihedron Newton yang disekutukan dengan polinomial terbitan. Polinomial berdarjah lima yang dipertimbangkan berbentuk $f(x, y) = ax^5 + bx^4 y + cx^3 y^2 + dx^2 y^3 + exy^4 + my^5 + nx + ty + k$.

Kata kunci: Hasil tambah eksponen, kekardinalan, saiz $p$-adic, polihedron Newton

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1.0 INTRODUCTION

In our discussion, we use notation $\mathbb{Z}_p$, $\Omega_p$, and $\text{ord}_p x$ to denote respectively the ring of $p$-adic integers, completion of the algebraic closure of $\mathbb{Q}_p$, the field of rational $p$-adic numbers and the highest power of $p$ which divides $x$. For each prime $p$, let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be an $n$-tuple polynomials in $\mathbb{Z}_p[x]$ where $\mathbb{Z}_p$ is the ring of $p$-adic integers and $x = \{x_1, x_2, \ldots, x_n\}$.

The estimation of $|V|$ has been the subject of many research in number theory one of which is in finding the best possible estimates to multiple exponential sums of the form $S(f;q) = \sum_{x \text{ mod } q} \exp \left( \frac{2 \pi i f(x)}{q} \right)$ where $f(x)$ is a polynomial in $\mathbb{Z}_p[x]$ and the sum taken over a complete set of residues $x$ modulo a positive integer $q$.

Loxton and Vaughn [1] are among the researchers who investigated $S(f;q)$ where $f$ is a non-linear polynomial in $\mathbb{Z}_p[x]$ and they found that the estimate of $S(f;q)$ depends on the value of $|V|$ the number of common zeros of the partial derivatives of $f$ with respect to $x$ modulo $q$. By using this result, the estimate of $S(f;q)$ was found by them in terms of invariants related to $f$. In his quest to find a more explicit estimate of $S(f;q)$, Mohd Atan [2] began by investigating the sum associated with lower degree polynomials. He considered in particular the non-linear polynomial $f(x,y) = ax^3 + bx^2 y + cx + dy + e$ with coefficients in $\mathbb{Z}_p$. He found that the $p$-adic sizes for the zero $(\xi, \eta)$ of this polynomial is $\text{ord}_p \xi \geq \frac{1}{2}(\alpha - \delta)$ and $\text{ord}_p \eta \geq \frac{1}{2}(\alpha - \delta)$ with $\delta = \max\left\{\text{ord}_p 3a, \frac{3}{2} \text{ord}_p b\right\}$.

Later, Mohd. Atan and Abdullah [3] considered a cubic polynomial of the form $f(x,y) = ax^3 + bx^2 y + cxy^2 + dy^3 + kx + my + n$ and obtained the $p$-adic sizes for the root $(\xi, \eta)$ of this polynomial as $\text{ord}_p \xi \geq \frac{1}{2}(\alpha - \delta)$ and $\text{ord}_p \eta \geq \frac{1}{2}(\alpha - \delta)$ with $\delta = \max\left\{\text{ord}_p 3a, \text{ord}_p b, \text{ord}_p c, \text{ord}_p d, \text{ord}_p e\right\}$.

Subsequently, in 1997 Chan and Mohd. Atan [4] investigated a polynomial of a higher degree then the one considered above in $\mathbb{Z}_p[x,y]$ of the form $f(x,y) = ax^4 + bx^3 y + cx^2 y^2 + dxy^3 + ey^4 + mx + ny + k$ and showed that for $(\xi, \eta)$ a root of $f(x,y)$, $\text{ord}_p \xi \geq \frac{1}{3}(\alpha - \delta)$ and $\text{ord}_p \eta \geq \frac{1}{3}(\alpha - \delta)$ with $\delta = \max\left\{\text{ord}_p a, \text{ord}_p b, \text{ord}_p c, \text{ord}_p d, \text{ord}_p e\right\}$.

Heng and Mohd. Atan [5] determined the cardinality associated with the partial derivatives functions of the cubic form $f(x,y) = ax^3 + bx^2 y + cx + dy + e$. In their
work, they attempt to find a better estimate by looking at the maximum number of common zeros associated with the partial derivatives $f_x(x, y)$ and $f_y(x, y)$. A sharper result was obtained with $\delta$ similar to the one considered by Mohd. Atan [2]. However, results for two-variable polynomials of higher degrees are less complete.

In this paper, we will discuss a method of determining explicitly the $p$-adic sizes of the components $(\xi, \eta)$ a common root of partial polynomial of $f(x, y)$ in $\mathbb{Z}_p[x, y]$ of degree five. The polynomial that we consider in this paper is of the form

$$f(x, y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + mx^5 + nx + ty + k$$

where the dominant terms are complete.

Our approach entails examination of combinations of indicator diagrams associated with the Newton polyhedrons of $f_x$ and $f_y$.

2.0 NEWTON POLYHEDRON

In this section, we give a brief description of the polyhedron as developed by [6]. It is a two-variable analogue of the $p$-adic Newton polygon in single variable as developed by [7].

Definition 2.1:

Let $p$ be a prime and $f(x, y) = \Sigma a_{ij}x^iy^j$ a polynomial in $\mathbb{Z}_p[x, y]$. We map the term $T_{ij} = a_{ij}x^iy^j$ of $f(x, y)$ to the points $P_{ij} = (i, ord_\rho a_{ij})$ in the three dimensional Euclidean space and call this set of points Newton diagram of $f(x, y)$. Below is an example of a Newton diagram for a lower degree polynomial.

Example 2.1:

![Newton Diagram Example](image_url)

**Figure 1** Newton diagram of $f(x, y) = 3x^2 + 2xy - y^2 + 27$ with $p = 3$
Definition 2.2:
Let $p$ be a prime and $f(x, y) = \sum a_{ij} x^i y^j$ a polynomial in $\Omega_p[x, y]$. The Newton polyhedron of $f(x, y)$ is the lower convex hull of the Newton diagram of $f(x, y)$. It is the highest convex connected surface which passes through or below the points $P_{ij}$ in the Newton diagram of $f(x, y)$. If $a_{ij} = 0$ then the associated point is omitted, since it lies at infinity above the $i-j$ plane. Below is the Newton polyhedron associated with the polynomial in Example 2.1.

Example 2.2:

![Newton Polyhedron](image)

Figure 2  The Newton polyhedron of $f(x, y) = 3x^2 + 2xy - y^2 + 27$ with $p = 3$

Definition 2.3:
Let $(\mu_i, \lambda_i, 1)$ be the normalized upward-pointing normals to the faces $F_i$ of $N_p$, the Newton polyhedron of a polynomial $f(x, y)$ in $Q_p[x, y]$. We map $(\mu_i, \lambda_i, 1)$ to the points $(\mu_i, \lambda_i)$ in the $x-y$ plane. If $F_r$ and $F_s$ are adjacent faces in $N_p$, sharing a common edge, we construct the straight line joining $(\mu_r, \lambda_r)$ and $(\mu_s, \lambda_s)$. If $F_r$ has a common edge with the vertical face $F$ say in $N_p$, we construct the straight line segment joining $(\mu_r, \lambda_r)$ and the appropriate point at infinity that corresponds to the normal of $F$, that is the segment along a line with slope $-\alpha/\beta$. We call the set of lines so obtained the indicator diagram associated with the Newton polyhedron of $f(x, y)$ [2]. The indicator diagram associated with the Newton polyhedron in Example 2.2 is as shown in the following example.
Example 2.3:

\[ f(x, y) = 3x^2 + 2xy - y^2 + 27 \]

Figure 3  Indicator diagram associated with the polynomial \( f(x, y) = 3x^2 + 2xy - y^2 + 27 \) with \( p = 3 \)

### 3.0 \( p \)-ADIC ORDERS OF ZEROS OF A POLYNOMIAL

In 1986 Mohd. Atan and Loxton conjectured that to every point of intersection of the combination of the indicator diagrams associated with the Newton polyhedrons of a pair of polynomials in \( \mathbb{Z}_p[x, y] \) there exist common zeros of both polynomials whose \( p \)-adic orders correspond to this point [6]. The conjecture is as follows:

Conjecture

Let \( p \) be a prime. Suppose \( f \) and \( g \) are polynomials in \( \mathbb{Z}_p[x, y] \). Let \((\mu, \lambda)\) be a point of intersection of the indicator diagrams associated with \( f \) and \( g \). Then there are \( \xi \) and \( \eta \) in \( \Omega_p \) satisfying \( f(\xi, \eta) = g(\xi, \eta) = 0 \) and \( \text{ord}_p \xi = \mu \), \( \text{ord}_p \eta = \lambda \).

A special case of this conjecture was proved by Mohd. Atan and Loxton [6]. Sapar and Mohd. Atan [8] improved this result and it is written as follows:

Theorem 3.1

Let \( p \) be a prime. Suppose \( f \) and \( g \) are polynomials in \( \mathbb{Z}_p[x, y] \). Let \((\mu, \lambda)\) be a point of intersection of the indicator diagrams associated with \( f \) and \( g \) at the vertices or simple points of intersections. Then there are \( \xi \) and \( \eta \) in \( \Omega_p \) satisfying \( f(\xi, \eta) = g(\xi, \eta) = 0 \) and \( \text{ord}_p \xi = \mu \), \( \text{ord}_p \eta = \lambda \).

In Theorem 3.2 we give the \( p \)-adic sizes of common zeros of partial derivatives of the polynomial \( f(x, y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k \). First, we have the assertion as in Lemma 3.1. In this lemma and the theorem that follows,
\[ \alpha_1 = \frac{4b + 2\lambda_2 c}{4(5a + \lambda_2 b)}, \alpha_2 = \frac{4b + 2\lambda_1 c}{4(5a + \lambda_1 b)}, \text{with } \lambda_1, \lambda_2 \text{ zeros of } \]

\[ k(\lambda) = (10dm - 4e^2)\lambda^2 + (10cm - 2de)\lambda + 2ce - d^2. \]

We note that clearly \( \alpha_1 \neq \alpha_2 \) if \( \lambda_1 \neq \lambda_2 \).

**Lemma 3.1**

Suppose \( U, V \) in \( \Omega_p \) with \( U = x + \alpha_1 y \) and \( V = x + \alpha_2 y \). Let \( p > 5 \) be a prime, \( a, b, c, d, e \) and \( m \) in \( Z_p \), \( \delta = \max\{\text{ord}_p a, \text{ord}_p b, \text{ord}_p c, \text{ord}_p d, \text{ord}_p e, \text{ord}_p m\} \), \( \text{ord}_p a, \text{ord}_p b \geq \alpha > \delta \) and \( \text{ord}_p b^2 > \text{ord}_p ac \). If \( \text{ord}_p U = \frac{1}{4} \text{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} \), \( \text{ord}_p V = \frac{1}{4} \text{ord}_p \frac{s + \lambda_2 t}{5a + \lambda_2 b} \) and \( \text{ord}_p (10cm - 2de)^2 > \text{ord}_p (10dm - 4e^2)(2ce - d^2) \) then \( \text{ord}_p x \geq \frac{1}{4}(\alpha - \delta) \) and \( \text{ord}_p y \geq \frac{1}{4}(\alpha - \delta) \).

**Proof:**

From \( U = x + \alpha_1 y \) and \( V = x + \alpha_2 y \), we have

\[ x = \frac{\alpha_1 U - \alpha_1 V}{\alpha_2 - \alpha_1} \text{ and } y = \frac{U - V}{\alpha_1 - \alpha_2} \]

Then,

\[ \text{ord}_p x = \text{ord}_p (\alpha_1 V - \alpha_2 U) - \text{ord}_p (\alpha_1 - \alpha_2) \quad (1) \]

and

\[ \text{ord}_p y = \text{ord}_p (U - V) - \text{ord}_p (\alpha_1 - \alpha_2) \quad (2) \]

with \( \text{ord}_p (\alpha_1 - \alpha_2) = \text{ord}_p \left( \frac{2b^2 - 5ac}{2(5a + \lambda_1 b)(5a + \lambda_2 b)} (\lambda_2 - \lambda_1) \right) \)

and \( \lambda_2 - \lambda_1 = -\sqrt{(10cm - 2de)^2 - 4(10dm - 4e^2)(2ce - d^2)} \)

\[ 10dm - 4e^2 \]

Since \( \text{ord}_p (10cm - 2de)^2 > \text{ord}_p (10dm - 4e^2)(2ce - d^2) \), we have \( \lambda_1 \neq \lambda_2 \) and

\[ \text{ord}_p (\lambda_1 - \lambda_2) = \frac{1}{2} \text{ord}_p \left( \frac{2ce - d^2}{10dm - 4e^2} \right) \]
Hence, from (1) and (3),
\[
\operatorname{ord}_p x = \operatorname{ord}_p (\alpha_2 U - \alpha_1 V) - \operatorname{ord}_p \left( \frac{(2b^2 - 5ac)(\lambda_2 - \lambda_1)}{2(5a + \lambda_2 b)(5a + \lambda_2 b)} \right)
\]

Suppose \( \min \{ \operatorname{ord}_p \alpha_2 U, \operatorname{ord}_p \alpha_1 V \} = \operatorname{ord}_p \alpha_2 U \), we have
\[
\operatorname{ord}_p x \geq \operatorname{ord}_p U + \operatorname{ord}_p \frac{4b + 2\lambda_2 c}{4(5a + \lambda_2 b)} - \operatorname{ord}_p \left( \frac{(2b^2 - 5ac)(\lambda_2 - \lambda_1)}{2(5a + \lambda_2 b)(5a + \lambda_2 b)} \right)
\]

Thus, we obtain
\[
\operatorname{ord}_p x \geq \operatorname{ord}_p U + \operatorname{ord}_p (2b + \lambda_2 c) - \operatorname{ord}_p (2b^2 - 5ac) - \operatorname{ord}_p (\lambda_1 - \lambda_2) + \operatorname{ord}_p (5a + \lambda_1 b)
\]
That is,
\[
\operatorname{ord}_p x \geq \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + \operatorname{ord}_p (2b + \lambda_2 c) - \operatorname{ord}_p (2b^2 - 5ac) - \frac{1}{2} \operatorname{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \operatorname{ord}_p (5a + \lambda_1 b)
\]

since \( \operatorname{ord}_p U = \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} \)

Suppose \( \min \{ \operatorname{ord}_p 2b, \operatorname{ord}_p \lambda_2 c \} = \operatorname{ord}_p \lambda_2 c \). Since \( \operatorname{ord}_p b^2 > \operatorname{ord}_p ac \), we have
\[
\operatorname{ord}_p x \geq \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + \operatorname{ord}_p b - \operatorname{ord}_p ac - \frac{1}{2} \operatorname{ord}_p \frac{2ce - d^2}{10dm + 4e^2} + \operatorname{ord}_p (5a + \lambda_1 b)
\]

Suppose \( \min \{ \operatorname{ord}_p 5a, \operatorname{ord}_p \lambda_1 b \} = \operatorname{ord}_p \lambda_1 b \) and since \( \operatorname{ord}_p b^2 > \operatorname{ord}_p ac \), we have
\[
\operatorname{ord}_p x \geq \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} + \operatorname{ord}_p b - \operatorname{ord}_p b^2 - \frac{1}{2} \operatorname{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \operatorname{ord}_p \lambda_1 b
\]

Since \( \operatorname{ord}_p (10cm - 2de)^2 > \operatorname{ord}_p (10dm - 4e^2)(2ce - d^2) \), we find that
\[
\operatorname{ord}_p x \geq \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b} - \frac{1}{2} \operatorname{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{1}{2} \operatorname{ord}_p \frac{2ce - d^2}{10dm - 4e^2} = \frac{1}{4} \operatorname{ord}_p \frac{s + \lambda_1 t}{5a + \lambda_1 b}
\]
Suppose \( \min\{\text{ord}_p s, \text{ord}_p \lambda t\} = \text{ord}_p s \) and \( \min\{\text{ord}_p 5a, \text{ord}_p \lambda_1 b\} = \text{ord}_p \lambda_1 b \). Then,

\[
\text{ord}_p x \geq \frac{1}{4} \left( \text{ord}_p s - \text{ord}_p \lambda t \right) \\
\geq \frac{1}{4} \left( \text{ord}_p s - \text{ord}_p a \right)
\]

By hypothesis,

\[
\text{ord}_p x \geq \frac{1}{4} (\alpha - \delta)
\]

Now from (2) and (3), we have

\[
\text{ord}_p y = \text{ord}_p (U - V) - \text{ord}_p \left( \frac{(2b^2 - 5ac)(\lambda_2 - \lambda_1)}{2(5a + \lambda_1 b)(5a + \lambda_2 b)} \right)
\]

Suppose \( \min\{\text{ord}_p U, \text{ord}_p V\} = \text{ord}_p U \) and since \( \text{ord}_p (5a + \lambda_1 b) = \text{ord}_p (5a + \lambda_2 b) \), we obtain

\[
\text{ord}_p y > \text{ord}_p U - \text{ord}_p \left( 2b^2 - 5ac \right) - \text{ord}_p (\lambda_2 - \lambda_1) + 2\text{ord}_p (5a + \lambda_1 b)
\]

That is,

\[
\text{ord}_p y \geq \frac{1}{4} \left( \text{ord}_p s + \lambda t \right) - \text{ord}_p ac - \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + 2\text{ord}_p (5a + \lambda_1 b) \\
= \frac{1}{4} \text{ord}_p (s + \lambda t) - \text{ord}_p ac - \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{7}{4} \text{ord}_p (5a + \lambda_1 b)
\]

Suppose \( \min\{\text{ord}_p 5a, \text{ord}_p \lambda_1 b\} = \text{ord}_p \lambda_1 b \). Since \( \text{ord}_p b^2 > \text{ord}_p ac \), we have

\[
\text{ord}_p y \geq \frac{1}{4} \text{ord}_p (s + \lambda t) - \text{ord}_p b^2 - \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{7}{4} \text{ord}_p b \\
+ \frac{7}{4} \left( \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} \right) \\
\geq \frac{1}{4} \text{ord}_p (s + \lambda t) - \frac{1}{4} \text{ord}_p b - \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} + \frac{1}{2} \text{ord}_p \frac{2ce - d^2}{10dm - 4e^2} \\
\geq \frac{1}{4} \text{ord}_p (s + \lambda t) - \text{ord}_p b
\]
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By hypothesis,
\[ \text{ord}_p y \geq \frac{1}{4} (\alpha - \delta) \]

We will get the same result if
\[
\min\{\text{ord}_p U, \text{ord}_p V\} = \text{ord}_p V, \quad \min\{\text{ord}_p 2b, \text{ord}_p \lambda_2 c\} = \text{ord}_p \lambda_2 c,
\]
\[
\min\{\text{ord}_p 5a, \text{ord}_p \lambda_1 b\} = \text{ord}_p a \quad \text{and} \quad \min\{\text{ord}_p s, \text{ord}_p \lambda_1 t\} = \text{ord}_p \lambda_1 t
\]

Therefore, we have
\[
\text{ord}_p x \geq \frac{1}{4} (\alpha - \delta) \quad \text{dan} \quad \text{ord}_p y \geq \frac{1}{4} (\alpha - \delta)
\]
as asserted.

**Theorem 3.2**

Let $f(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + my^5 + nx + ty + k$ be a polynomial in $\mathbb{Z}_p[x,y]$ with $p > 5$. Suppose $\alpha > 0$, $\delta = \max\{\text{ord}_p a, \text{ord}_p b, \text{ord}_p c, \text{ord}_p d, \text{ord}_p e, \text{ord}_p m\}$, $\text{ord}_p b^2 > \text{ord}_p ac$ and $\text{ord}_p (10c - 2de) > \text{ord}_p (10dm - 4e^2)(2ce - d^2)$.

If $\text{ord}_p f_x (0,0), \text{ord}_p f_y (0,0) \geq \alpha > \delta$ there exists $(\xi, \eta)$ in $\Omega_p^2$ such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $\text{ord}_p \xi \geq \frac{1}{4} (\alpha - \delta), \text{ord}_p \eta \geq \frac{1}{4} (\alpha - \delta)$.

Proof:

Let $g = f_x$ and $h = f_y$, and $\lambda$ a constant.

Then,
\[
(g + \lambda h)(x,y) = \left(5a + \lambda b\right)x^4 + \left(4b + 2\lambda c\right)x^3 y + \left(3c + 3\lambda d\right)x^2 y^2 + \left(2d + 4\lambda e\right)xy^3 + \left(e + 5\lambda m\right)y^4 + \lambda \lambda t
\]

and
\[
\frac{(g + \lambda h)(x,y)}{5a + \lambda b} = x^4 + \left(\frac{4b + 2\lambda c}{5a + \lambda b}\right)x^3 y + \left(\frac{3c + 3\lambda d}{5a + \lambda b}\right)x^2 y^2 + \left(\frac{2d + 4\lambda e}{5a + \lambda b}\right)xy^3 + \left(\frac{e + 5\lambda m}{5a + \lambda b}\right)y^4 + \frac{s + \lambda t}{5a + \lambda b}
\]

(4)

Let $\alpha_{ij}$ denote the coefficients of $x^iy^j$ in the completed quartic form of Equation (4), $0 \leq i \leq 4$, $0 \leq j \leq 4$. By completing the quartic Equation (4) and by solving simultaneously equations $\alpha_{ij}(\lambda) = 0$, $i \neq 0$, $j \neq 0$, and $i + j = 4$, we obtain
where \( \lambda \) satisfies the equation
\[
\frac{e + 5\lambda m}{5a + \lambda b} - \frac{1}{2 \frac{(d + 2\lambda e)^2}{(c + \lambda d)(5a + \lambda b)}} = 0
\]
That is,
\[
(10dm - 4e^2)\lambda^2 + (10cm - 2de)\lambda + 2ce - d^2 = 0
\]  
(6)
From (6), we have two values of \( \lambda \), say \( \lambda_1, \lambda_2 \) where
\[
\lambda_1 = \frac{-\left(10cm - 4e^2\right) + \sqrt{(10cm - 2de)^2 - 4\left(10dm - 4e^2\right)(2ce - d^2)}}{2\left(10dm - 4e^2\right)}
\]
and
\[
\lambda_2 = \frac{-\left(10cm - 4e^2\right) - \sqrt{(10cm - 2de)^2 - 4\left(10dm - 4e^2\right)(2ce - d^2)}}{2\left(10dm - 4e^2\right)}
\]
\( \lambda_1 \neq \lambda_2 \), because \( ord_p(10cm - 2de)^2 > ord_p (10dm - 4e^2)(2ce - d^2) \).
Now, let
\[
U = x + \frac{4b + 2\lambda c}{4(5a + \lambda b)}y
\]  
(7)
\[
V = x + \frac{4b + 2\lambda c}{4(5a + \lambda b)}y
\]  
(8)
\[
F(U,V) = (g + \lambda_1 h)(x,y)
\]  
(9)
and
\[
G(U,V) = (g + \lambda_2 h)(x,y)
\]  
(10)
By substituting \( U \) and \( V \) in (5), we obtain a polynomial in \( (U,V) \) as follows:
\[
F(U,V) = (5a + \lambda_1 b)U^4 + s + \lambda_1 t
\]  
(11)
\[
G(U,V) = (5a + \lambda_2 b)V^4 + s + \lambda_2 t
\]  
(12)
The combination of the indicator diagrams associated with Newton polyhedron of (11) and (12) is as shown below.
ESTIMATION OF $p$-ADIC SIZES OF COMMON ZEROS OF PARTIAL DERIVATIVE

From Figure 4 and Theorem 3.1 there exists $(\hat{U}, \hat{V})$ in $\Omega^2_p$ such that $F(U, V) = (5a + \lambda_1 b)U^4 + s + \lambda_1 t$ and $G(U, V) = (5a + \lambda_2 b)U^4 + s + \lambda_2 t$.

From Figure 4 and Theorem 3.1 there exists $(\hat{U}, \hat{V})$ in $\Omega^2_p$ such that $F(U, V) = 0$, $G(U, V) = 0$ and $ord_p U = \mu_1$, $ord_p V = \mu_2$ with $\mu_1 = \frac{1}{4} ord_p \frac{s + \lambda_1 t}{5a + \lambda_1 b}$ and $\mu_2 = \frac{1}{4} ord_p \frac{s + \lambda_2 t}{5a + \lambda_2 b}$. Let $U = \hat{U}$ and $V = \hat{V}$ in (7) and (8). There exists $(x_0, y_0)$ in $\Omega^2_p$ such that

$$x_0 = \frac{\alpha_2 \hat{U} - \alpha_1 \hat{V}}{\alpha_2 - \alpha_1} \text{ and } y_0 = \frac{\hat{U} - \hat{V}}{\alpha_1 - \alpha_2}.$$

Hence, $ord_p x_0 = ord_p (\alpha_2 \hat{V} - \alpha_1 \hat{U}) - ord_p (\alpha_1 - \alpha_2)$ and $ord_p y_0 = ord_p (V - U) - ord_p (\alpha_1 - \alpha_2)$.

From Lemma 3.1, we find that $ord_p x_0 \geq \frac{1}{4}(\alpha - \delta)$ and $ord_p y_0 \geq \frac{1}{4}(\alpha - \delta)$. Let $\xi = x_0$ and $h = y_0$. By back substitution in (9) and (10) and since $\lambda_1 \neq \lambda_2$ we have $g(\xi, \eta) = f_x(\xi, \eta) = 0$ and $h(\xi, \eta) = f_y(\xi, \eta) = 0$. Thus, $ord_p \xi = ord_p x_0 \geq \frac{1}{4}(\alpha - \delta)$ and $ord_p \eta = ord_p y_0 \geq \frac{1}{4}(\alpha - \delta)$ with $(\xi, \eta)$ a common zero of $g$ and $h$.

4.0 CONCLUSION

Our investigation observes that if $p$ is an odd prime, $p > 5$, $f(x, y) = ax^5 + bx^4 y + cx^3 y^3 + dx^2 y^3 + ey^4 + my^5 + nx + ty + k$ a polynomial in $Z_p[x, y]$ with $ord_p b^2 > ord_p ac$ and $ord_p (10cm - 2de)^2 > ord_p (10dm - 4e^2)(2ce - d^2)$, then the $p$-adic sizes of common zeros of partial derivatives of this polynomial is
\[ \text{ord}_p \xi \geq \frac{1}{4} (\alpha - \delta) \text{and } \text{ord}_p \eta \geq \frac{1}{4} (\alpha - \delta) \]

with \( \xi = \max\{\text{ord}_p a, \text{ord}_p b, \text{ord}_p c, \text{ord}_p d, \text{ord}_p e, \text{ord}_p m\} \) and \( \text{ord}_p f(x,0), \text{ord}_p f(0,0) \geq \alpha > \xi. \)

This work demonstrates that common zeros of certain \( p \)-adic orders of partial derivatives of a two-variable polynomial with coefficients in \( \mathbb{Z}_p \) can be obtained through applications of the Newton polyhedron technique. We have also shown that the \( p \)-adic orders of the zeros can be determined explicitly in terms of the \( p \)-adic orders of the coefficients of the dominant terms of the two-variable polynomial. This work extends future direction in finding explicit estimates of exponential sums associated with much higher degree of two-variable polynomials, which will in turn pave the way to finding better estimates of the sum associated with polynomials in several variables.

REFERENCES