Numerical Modeling of Blood Flow in Irregular Stenosed Artery with the Effects of Gravity

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For the Cardiovascular Diseases, the Familiar Ones Such as Cardiovascular Diseases, the familiar ones such as hypertension and hyperlipidemia, get a rise in blood pressure and blood viscosity at lower levels of gravitational force. These may cause the coronary blood flow to be insufficient to arrive, which leads to myocardial ischemia. Hypertension and hyperlipidemia are risk factors that contribute to atherosclerosis, which is a major cause of coronary heart disease. Atherosclerosis is a disease that occurs when the lining of an artery (the innermost layer of the artery) becomes thick and rigid, causing the artery to narrow and become less flexible. This can lead to atherosclerosis, which can cause a heart attack if the blockage affects the coronary arteries.

The cardiovascular system, which consists of the heart and blood vessels, which plays an important role in transportation, protection, and regulation of the human body. The pressure gradient is produced as the heart pumps so that blood will flow through vessels throughout the body (Ku [1]). Among the cardiovascular diseases, the familiar ones such as stroke and atherosclerosis are closely related to abnormality, disorder and malfunction of blood flow characteristics in human body. Due to this, blood flow related problems has obtained significant interest by biomedical researchers.

Segmental narrowing of an artery due to substances deposition or intravascular plaques is called an arterial stenosis (Mandal et al. [2]). This may be caused by unhealthy living conditions such as exposure to tobacco smoke, lack of physical activity, and so on. It is always followed by serious changes in blood flow, pressure distribution, wall shear stress, and flow resistance. Once arterial stenosis occurs, atherosclerotic plaques would protrude into lumen of blood vessels. Consequently, resistance is increased; hence blood flow is insufficient to reach every cells and this resists nutrient supplement. These could lead to widespread of health disorders which may then worsen to various illnesses. To more serious extent, these abnormalities in blood flow could contribute substantial fatal health risks. Several theoretical and experimental studies by mean of blood flow characteristics with arterial stenosis were done, such as Ling and Atabek [3], Padmanabhan [4], Back et al. [5], Johnston and Kilpatrick [6], Chakravarty and Sannigrahi [7], Ku [1], Jung et al. [8], Mandal [9] and Mustapha et al. ([10], [11], [12]).

Gravitiation is a natural fact such that physical bodies are draw to the earth due to masses. Studies suggested that gravity force is one of the bases of regulating blood flow. For space programmes to be carried out safely, studies on physical changes during absence of weight are performed. In space activities, astronauts encounter a condition of microgravity, causing body fluids to distribute more to upper parts of the body which is very different from the condition with gravitation on earth (Kim et al. [13]). However at launch and on return, hypergravity is frequently faced. These types of changes in blood flow velocity due to gravitational force may cause several health problems especially when there are large changes in gravitational force.
unhealthy deposits in blood vessels (Payne [14]; Burrowes et al. [15], [16]). Payne [14] carried out the analysis of the effects of gravity and wall thickness on blood flow behavior. Navier-Stokes equations were derived with additional of gravitational term, \( gS \), where \( g \) is the gravitational acceleration and \( S \) is the vessel slope. The equations are linked to a simple vessel wall model. The result showed that amplitude of the velocity pulse changes with slope.

On the other hand, it is noticed that the acceleration of gravity does not vary only during space activities; even on the earth itself, there are various causes affecting gravitation, such as latitude, altitude, tidal effects, and so forth. For example, at the equator, gravity acceleration gives a value of 9.78\( \text{m/s}^2 \) but it becomes 9.832\( \text{m/s}^2 \) when shifted to the poles (Boynton [17]). Besides, during postural changes, blood flow distribution and pressure in human body will also be influenced. When a head-up tilt (a test operated to diagnose patients suffered from certain level of dizziness and syncope) is performed, the patient’s blood assembles at lower limbs because of gravity during the tilt. This will lead to increment of blood pressure in the lower body and decrement in the head (Heusden et al. [18]; Olufsen et al. [19]).

Mathematical modelling on blood flow in stenosed arteries has been expected to be important in studies of arteriosclerotic plaques and the effects. Various studies formulated an unsteady nonlinear two-dimensional model. It was observed that blood behaves as Newtonian fluid when it flows through wider arteries. But when flowing through a narrower artery at which shear stress is lower, it behaves like a non-Newtonian fluid (Chakravarty and Mandal [20]). In this study, blood flow in arteries is modelled by axisymmetric Navier-Stokes equations; while the artery itself is assumed to be an elastic cylindrical tube composing of Newtonian fluid which is the blood. Analyses of blood flow characteristics under the effects of gravity are taken into consideration. The geometry of stenosis chosen is an irregular stenosis in cylindrical artery vessels following data given by Back et al. [5], which mimics the roughness of real constricted blood vessel surface. Blood flow is assumed to be unsteady, two-dimensional, and incompressible. Walls of artery vessels are considered to be elastic and axisymmetric. The numerical method chosen in this study is the finite difference relaxation (SOR) method following a MATLAB coding accomplished by Mustapha et al. [11]. Successive over-relaxation (SOR) method is involved to solve the pressure-Poisson equation.

### 2.0 STENOSIS MODEL

In this study, the profile of stenosis model is chosen to be an irregular stenosis following data developed by Back et al. [5] which imitates the real blood vessel surface roughness.

![Figure 1: Geometry of the irregular stenosis](image)

#### 3.0 GOVERNING EQUATIONS

In this study, streaming blood in the constricted artery segment is accounted to be incompressible, unsteady, laminar, fully-developed, and is modeled as Newtonian fluid. The fundamental governing equations of the studies comprise of the continuity equations and the momentum equations. Conservative forms of the governing equations are presented as follow:

\[
\frac{-\partial w}{r \partial z} + \frac{\partial \left(u^r\right)}{\partial r} = 0
\]

\[
\frac{\partial u}{\partial t} + \frac{\partial \left(wu\right)}{\partial r} + \frac{\partial \left(w^r\right)}{\partial z} + \frac{u}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \left[u \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r} \right]}{r}
\]

\[
\frac{\partial w}{\partial t} + \frac{\partial \left(wu\right)}{\partial r} + \frac{\partial \left(w^r\right)}{\partial z} + \frac{w}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial \left[u \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} + g \cos \theta\right]}{r}
\]
\[ \begin{align*}
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{u}) &= \nabla \cdot (\mu \nabla \mathbf{u}) + \nabla p - \mathbf{f} \\
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial p}{\partial x} + \mu \frac{\partial u}{\partial x} \right) + f_x \\
\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial p}{\partial y} + \mu \frac{\partial v}{\partial y} \right) + f_y \\
\rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial z} &= \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + \mu \frac{\partial w}{\partial z} \right) + f_z
\end{align*} \]

where \( \mathbf{f} \) is the body force per unit mass, \( \mu \) is the dynamic viscosity of the fluid, and \( p \) is the pressure. The last term on the right-hand side of each equation represents the body forces per unit mass acting on the fluid, i.e., \( \mathbf{f} = \rho \mathbf{f} \).
Discretization of the continuity equation takes place at the location of pressure, \( p \), which is also the centre of cell. Its discretized form at the \((i, j)\) th cell is:

\[
x_i R_i \left( \frac{w_i - w_{i-1}}{\Delta z} \right) + \left( x_i u_i - x_i u_{i-1} \right) \frac{\Delta t}{\Delta x} - (x_i) \left( \frac{\partial R}{\partial z} \right)_i \left( \frac{w_i - w_{i+1}}{\Delta x} \right) = 0
\]

where

\[
w_w = \frac{w_i + w_{i+1} + w^r_{i-1} + w^l_{i+1}}{4},
\]

\[
w_a = \frac{w^r_i + w^l_i + w^l_{i-1} + w^r_{i+1}}{4},
\]

\[
x_0 = x_i - \frac{\Delta x}{2}, \quad R_i = R(z), \quad z_i = z - \frac{\Delta z}{2},
\]

with \((z_i, x_i)\) and \((z, x)\) represent the coordinates of the cell center and the cell faces respectively, as shown in Figure 2.

Next, the axial momentum equation is rearranged and expressed in finite difference form as

\[
\frac{w_{i+1} - w_i}{\Delta t} = 2 \left( \frac{P_{i+1} - P_i}{\Delta z + \Delta z_{i-1}} \right) + \frac{x_i \left( \frac{\partial R}{\partial z} \right)_i \left( P - P_i \right)}{\Delta x} + (wme)_{i, j} + \frac{\cos \theta}{Fr}
\]

with

\[
P_i = \frac{(P^l_i + P^r_{i-1}) \Delta z_{i-1} + (P^l_i + P^r_{i+1}) \Delta z_i}{2(\Delta z + \Delta z_{i-1})},
\]

\[
P_i = \frac{(P^l_i + P^r_{i-1}) \Delta z_{i-1} + (P^l_i + P^r_{i+1}) \Delta z_i}{2(\Delta z + \Delta z_{i-1})},
\]

\[
\text{wme}_{i, j} = \text{Conv}^w_{i, j} + \frac{1}{Re} (\text{Diff}^w_{i, j}).
\]

where \(\text{Conv}^w_{i, j}\) and \(\text{Diff}^w_{i, j}\) describes the convective and diffusive terms of the axial momentum equation at \( k \)-th time level at the \((i, j)\) th cell. The discretized expressions for these terms are:

\[
\text{Conv}^w_{i, j} = \frac{x_i}{R_i} \left( \frac{\partial R}{\partial z} \right)_i \left[ \left( 1 - \beta \right) \frac{w^r_{i+1} - w^l_{i+1}}{\Delta z} \right] - \left[ \left( 1 - \beta \right) \frac{w^2_{i+1} - w^2_i}{\Delta z} + \beta \frac{w^i_{i+1} - w^i_{i-1}}{\Delta z} \right] - \left( 1 - \beta \right) \frac{w^2_{i+1} - w^2_i}{\Delta x} + \beta \frac{w^i_{i+1} - w^i_{i-1}}{\Delta x}
\]

\[
\text{Diff}^w_{i, j} = \frac{1}{R_i} \left[ \left( 1 + \frac{2x_i}{s_i} \right) \left( \frac{\partial R}{\partial z} \right)_i \right] \left( w^2_{i+1} - 2w^2_i + w^2_{i-1} \right) + 2 \left( \frac{s_i}{x_i} \Delta t \right) - \left[ \left( 1 - \beta \right) \frac{w^2_{i+1} - w^2_i}{\Delta z} + \beta \frac{w^i_{i+1} - w^i_{i-1}}{\Delta z} \right] - \left( 1 - \beta \right) \frac{w^2_{i+1} - w^2_i}{\Delta x} + \beta \frac{w^i_{i+1} - w^i_{i-1}}{\Delta x}
\]

The convective term as in equation (29) is differenced with a combination of central differencing and second order upwind schemes. As seen equation (29), a combination factor, \(\beta\), is introduced. It is determined from numerical stability. When \(\beta = 0\), the scheme becomes central differencing and when \(\beta = 1\), the scheme approaches second order upwind differencing.

For the axial momentum equation, the differential symbols are defined by:

\[
w_r = \frac{w^r_i + w^r_{i-1}}{2} \quad w_l = \frac{w^l_i + w^l_{i+1}}{2} \quad w = \frac{w^r_i + w^l_{i+1}}{2} \quad w^l = \frac{w^l_i + w^l_{i+1}}{2} \quad u = \frac{u^r_i + u^l_{i+1}}{2} \quad u^l = \frac{u^l_i + u^l_{i+1}}{2} \quad u^r = \frac{u^r_i + u^r_{i-1}}{2}
\]

where the suffixes \(r, l, t\) and \(b\) represents right, left, top and bottom middle positions of the cell faces while suffix \(m\) represents the middle of the cell faces.

Then, the momentum fluxes, \(\phi\), for axial momentum are expressed as:

\[
\text{if } w_r \geq 0 \quad \phi_r = w^r_i \quad \text{if } w_r < 0 \quad \phi_r = w^l_i
\]
if \( w_i > 0 \), \( \phi_{ai} = w_i^i \); if \( w_i < 0 \), \( \phi_{ai} = w_i^i \).

Similarly,

if \( u_i > 0 \), \( \phi_{ai} = w_i^i \); if \( u_i < 0 \), \( \phi_{ai} = w_i^i \).

For radial momentum equation, the finite difference form is expressed as:

\[
\frac{u_{i,j}^{t+1} - u_{i,j}^{t}}{\Delta t} = \frac{1}{R_i^2} \left( \frac{P_i^t - P_{i,j}^t}{\Delta x} + um_{e,i}^j \right)
\]

(31)

where

\[
ume_{i,j} = \text{Con} u_{i,j} + \frac{1}{Re} \left( \text{Diff} u_{i,j} \right).
\]

(32)

Con\( u_{i,j} \) and Diff\( u_{i,j} \) are convective and diffusive terms of the radial momentum equation at \( k \)-th time level at the \((i,j)\)th cell. The terms are differentiated in the same manner as in the axial momentum equation where

\[
\text{Con} u_{i,j} = \frac{x}{R_i^2} \frac{\partial R_i^2}{\partial z_{ij}} \left[ \left( 1 - \beta \right) w_{ui,k} - w_{ui,j} \right]
\]

\[
+ \frac{\beta w_{uk} - w_{uk}}{\Delta x} + \frac{x_j}{R_i^2} \frac{\partial^2 u_{i,j}^t}{\partial x_j^2}
\]

\[
- \frac{1}{R_i^2} \left( 1 - \beta \right) u_{i,j}^t \frac{u_{i,j}^t - u_{i,j}^t}{\Delta x} - u_{i,j}^t \frac{u_{i,j}^t}{x_i R_{i,j}^2}
\]

\[
\text{Diff} u_{i,j} = \frac{1}{R_i^2} \left( \frac{\partial R_i^2}{\partial z_{ij}} \right) \left[ \left( 1 + x_j \right) \left( \frac{\partial R_i^2}{\partial x_j} \right) \right] u_{i,j}^t - 2u_{i,j}^t + u_{i,j}^t
\]

\[
+ \frac{u_{i,j}^t - u_{i,j}^t}{2\Delta x} + 2x_j \left( \frac{\partial R_i^2}{\partial z_{ij}} \right) u_{i,j}^t - x_j \left( \frac{\partial R_i^2}{\partial x_j} \right)
\]

\[
- x_j \left( \frac{\partial^2 R_i^2}{\partial x_j^2} \right) + \left( R_i^2 \right) \left( \frac{\partial^2 R_i^2}{\partial z_{ij}^2} \right) \frac{x_j}{x_i^2}
\]

(33)

Next, the Poisson equation for pressure is gained by coupling and combining the discretized form of the continuity and momentum equations. After rearranging, the final form of Poisson equation for pressure would be:

\[
\frac{\Delta k}{\Delta t} \frac{P_{i,j}^{k+1} - P_{i,j}^k}{\Delta t} = A_{i,j} u_{i,j}^k + B_{i,j} u_{i,j}^k + C_{i,j} P_{i,j}^k + D_{i,j} P_{i,j}^{k+1} + E_{i,j} P_{i,j}^{k+1} + F_{i,j} P_{i,j}^{k+1} + G_{i,j} P_{i,j}^{k+1} + H_{i,j} P_{i,j}^{k+1} + I_{i,j} P_{i,j}^{k+1} + J_{i,j} P_{i,j}^{k+1} + K_{i,j} P_{i,j}^{k+1} + L_{i,j} P_{i,j}^{k+1} + M_{i,j} P_{i,j}^{k+1} + N_{i,j} P_{i,j}^{k+1} + O_{i,j} P_{i,j}^{k+1} + P_{i,j}^{k+1} + Q_{i,j} P_{i,j}^{k+1} + R_{i,j} P_{i,j}^{k+1} + S_{i,j} P_{i,j}^{k+1} + T_{i,j} P_{i,j}^{k+1} + U_{i,j} P_{i,j}^{k+1} + V_{i,j} P_{i,j}^{k+1} + W_{i,j} P_{i,j}^{k+1} + X_{i,j} P_{i,j}^{k+1} + Y_{i,j} P_{i,j}^{k+1} + Z_{i,j} P_{i,j}^{k+1} + a_{i,j} \Delta P_{i,j}
\]

(35)

Here, \( \text{Div} \) represents the discretized form of divergence of velocity field at the \((i,j)\)th cell and are described as:

\[
\text{Div}_{i,j} = \frac{u_{i,j} - u_{i,j}^t}{\Delta x}
\]

(36)

Then, the expressions for \( A, B, C, D, E, F, G, H, S \) are given as follows:

\[
A_{i,j} = \frac{x_i R_i^2}{\Delta x} + \frac{x_i R_i^2}{\Delta x} + \frac{R_i^2 \Delta x}{R_i^2 \Delta x}
\]

(37)

The next step is to solve the Poisson equation for pressure (35) by the successive over-relaxation (SOR) method with a certain number of iterations. This is to get the intermediate pressure field at the \( l \)th time step. Here, the value of the over-relaxation parameter is taken to be 1.2.

6.1 Pressure and Velocity Corrections

After solving the Poisson equation of pressure, the pressure obtained is an intermediate hence inaccurate. The velocities obtained do not satisfy the continuity equation. So, pressure and velocities should undergo a correction stage to get better accuracy. Here, a pressure-correction relation is introduced:

\[
P_{i,j} = P_{i,j}^t + \alpha \Delta P_{i,j}
\]

(38)
where $P'_{ij}$ is the intermediate pressure obtained from the Poisson equation of pressure, $\omega_j$ (which is $\leq 0.5$) is an under relaxation parameter and $\Delta P_{ij}$ is the pressure error term described as

$$\Delta P_{ij} = \frac{Div'_{ij}}{\Delta A_{ij}}$$  \hspace{1cm} (39)

where $Div'_{ij}$ is the divergence value of velocity field at the $(i,j)$ th cell. Next, the velocity correction formulas are

$$w'^{i+1}_{ij} = w^i_{ij} + \frac{\Delta t \Delta P_{ij}}{0.5(\Delta z^i_{ij} + \Delta z_{ij})},$$  \hspace{1cm} (40)

$$w'^{i+1}_{i,j+1} = w^{i+1}_{i,j} - \frac{\Delta t \Delta P_{ij}}{0.5(\Delta z^i_{ij} + \Delta z_{ij})},$$  \hspace{1cm} (41)

$$u'^{i+1}_{ij} = u^i_{ij} + \frac{\Delta t \Delta P_{ij}}{R_e \Delta t},$$  \hspace{1cm} (42)

and $u'^{i+1}_{i,j+1} = u^{i+1}_{i,j} - \frac{\Delta t \Delta P_{ij}}{R_e \Delta t}$  \hspace{1cm} (43)

where $w^i_{ij}$, $w'^{i+1}_{ij}$, $u^i_{ij}$ and $u'^{i+1}_{ij}$, are the updated velocity components obtained after solving the Poisson equation of pressure.

### 6.2 Stability Restriction

For the numerical computation to be stable, several considerations need to be made. Restrictions on the mesh sizes, $\Delta z$ and $\Delta x$, and time interval $\Delta t$ are imposed. Markkan and Proctor [26] suggested that, with relation to fluid convection, the fluid cannot pass through more than one cell in each time step. So the time step must obey the following inequality:

$$\Delta t \leq \text{Min} \left[ \frac{\Delta z}{|\nu|}, \frac{\Delta x}{|\nu|} \right].$$  \hspace{1cm} (44)

As suggested by Welch et al. [27], momentum must not diffuse more than one cell in each time step. The second stability restriction, which is related to the viscous effects of fluid, is suggested by Hirt [28] such that

$$\Delta t \leq \text{Min} \left[ \frac{Re \Delta x \Delta z^2}{2(\Delta x^2 + \Delta z^2)}, \Delta x \right].$$  \hspace{1cm} (45)

Combining and simplifying inequalities (44) and (45), the time step should be concluded to obey the following inequality:

$$\Delta t = c \text{Min} \left[ \Delta t_1, \Delta t_2 \right]$$  \hspace{1cm} (46)

where the constant $c$ (lying between 0.2 and 0.5) is added as a considerable computational saving.

Other than that, the combination factor, $\beta$, should be picked according to the following inequality:

$$1 \geq \beta \geq \text{Max} \left[ \frac{\nu \Delta t}{\Delta x^2}, \frac{u \Delta t}{\Delta z^2} \right]$$  \hspace{1cm} (47)

However, this inequality yields only a very small and considered not so significant. It can be improved by multiplying a factor of 1.2 as a safety measure.

### 7.0 RESULTS AND DISCUSSIONS

The imposed numerical algorithm based on Marker and Cell (MAC) method is validated by Mustapha et al. ([10]-[12]) using Matlab Programming software. In order to perform numerical computations of the desired quantities of major physiological significance, the following parameter values have been made use of:

$$\rho = 1.06 \times 10^3 \text{ kg m}^{-3} \text{; } \omega = 2\pi f_r \text{; } f_r = 1.2 \text{Hz} \text{; } \theta = 0^\circ \text{; } U_m = 0.5 \text{; } Re = 50$$

The numerical results presented in graphs are shown in this section.

#### 7.1 Streamlines For Different Values Of Froude Numbers

![Figure 3 Patterns of streamlines for Fr=0.25](image)

![Figure 4 Patterns of streamlines for Fr=0.5](image)

![Figure 5 Patterns of streamlines for Fr=1](image)

Figures 3-5 show the patterns of streamlines regarding blood flow through the constricted arterial segment with single irregular stenosis. Variation of the dimensionless Froude number represents values of gravitational force where lower value of Froude number indicates higher value of gravity. From the streamlines plotted, it is noticed that the streamlines do not overlap. This means that at each point there
is only one velocity. No recirculation region is observed for these 3 values of Froude numbers with this mild stenosis.

### 7.2 Wall Shear Stress

![Wall Shear Stress Graph](image)

**Figure 6** Variations of wall shear stress at different values of Froude numbers

Variations of wall shear stress comparing different values of Froude numbers along the dimensionless axial position are demonstrated in Figure 4. Wall shear stress is influenced by different values of gravity. It is clearly noticed that at lower Froude number (meaning higher gravitational force), wall shear stress is relatively higher.

### 7.3 Axial Velocity

![Axial Velocity Graph](image)

**Figure 7** Axial velocity profiles for different values of Froude numbers at $z = 20.16$

Figure 5 illustrates how gravity influences axial velocity profiles of the constricted arterial region. The velocity profiles are analyzed at position $z = 20.16$ which is one of the critical heights of the stenosis. The velocities decrease from their individual maximum. Comparing the graphs, axial velocity for $Fr=0.25$ (highest relative gravitational force) is noticed to have the relatively highest individual maxima, and vice versa. Hence, it can be concluded that axial velocity rises upon increment of gravity.

### 8.0 CONCLUSION

An unsteady two dimensional nonlinear model of blood flow through an irregular stenosed arterial segment is developed to study the effects of gravitation on blood flow. Blood is considered as an incompressible Newtonian fluid. The governing equations are non-dimensionalised and transformed using radial transformation. Then, they are solved using the finite difference approximations based on Marker and Cell (MAC) method. Values of gravitation is described dimensionlessly via different values of Froude number. Lower value of Froude number represents a condition with higher gravitational force and vice versa.

Streamlines of blood flow through the constricted arterial segment are plotted (Figures 3-5) to analyse differences of the flow patterns. Variations of wall shear stress along the constricted region is also plotted (Figure 6). Then in Figure 7, axial velocity profiles for different values of Froude numbers are plotted. As a summary, wall shear stress and axial velocity at lower Froude number (higher gravitational force) give higher values than that of a condition at higher Froude number.

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### References


