Hirota-Sato Formalism on Some Nonlinear Waves Equations

Noor Aslinda Ali*, Zainal Abdul Aziz², ³

¹Departement Sains Matematik, Fakulti Sains, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
²UTM Centre for Industrial and Applied Mathematics, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

*Corresponding author: zainalabdadz@gmail.com

Abstract

This article demonstrates that Hirota’s direct method or scheme for solving nonlinear waves equation is linked to Sato theory, and eventually resulted in the Sato equation. This theoretical framework or simply the Hirota-Sato formalism also reveals that the τ – function, which underlies the analytic form of soliton solutions of theses physically significant nonlinear waves equations, shall acts as the key function to express the solutions of Sato equation. From representation theory of groups, it is shown that the τ -- function in the bilinear forms of Hirota scheme are closely connected to the Plucker relations in Sato theory. Thus Hirota-Sato formalism provides a deeper understanding of soliton theory from a unified viewpoint. The Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera equations are used to verify this framework.

Keywords: Hirota-Sato Formalism; τ – function; Plucker relations; Kadomtsev-Petviashvili (KP) equation; Korteweg-de Vries (KdV) equation; Sawada-Kotera equation

1.0 INTRODUCTION

There are several methods of studying the integrable nonlinear waves equations that have soliton solutions, where each technique has its own suppositions and areas of usage. For example, the inverse scattering transform (IST) can be used to solve initial value problems, but it uses powerful analytical methods and quantum scattering theory (e.g. [1]), and therefore makes strong assumptions about the nonlinear equations. On the lesser extreme, one can find a travelling wave solution to almost all equations by a simple substitution which reduces the equation to an ordinary differential equation (e.g. [2]). Between these two extremes lies Hirota’s direct/bilinear method. Although the transformation was intrinsically inspired by IST, Hirota’s method does not need the same mathematical assumption and, as a consequence, the method is applicable to a wider class of equations than IST (e.g. [3]). At the same time, because it does not use such sophisticated techniques, it usually produces a smaller class of solutions, the multi-soliton solutions. It is particularly efficient for constructing multisoliton solutions to integrable nonlinear waves equations. The advantage of it over others is that it is algebraic rather than analytic. In many problems the key to further developments is a detailed understanding of soliton scattering, and in such cases Hirota’s method is the optimal tool.

Hirota’s method is an effective tool as it can be employed without a deep knowledge of the mathematics that lies beneath,
Hirota direct method was first introduced by Hirota in his framework, the beauty and power of Hirota-Sato formalism researcher of applied mathematics. Even in this preamble of the algebraic geometry (e.g. [4-5]). We do not attempt to reach quite Hirota's method is given by

\[ 2\mathbb{P}(D_x + D^2_x) \tau \cdot \tau = 0. \]  

The logarithmic (or dependent variable) transformation, via Hirota’s method is given by

\[ u = 2(\log \tau)_{xx}. \]  

The Hirota bilinear form of KdV is then given by

\[ D_x(D_x + D^2_x) \tau \cdot \tau = 0. \]  

c) The fifth-order Sawada-Kotera equation can be written as

\[ u_t + 45u^2u_x + 15(u_xu_{xx} + u_{xxx}u) + u_{xxxx} = 0. \]  

The logarithmic transformation is the same as in (2.5), i.e.

\[ u = 2(\log \tau)_{xx}. \]  

Thus, the Hirota bilinear form of Sawada-Kotera is of the form

\[ D_x(D_x + D^2_x) \tau \cdot \tau = 0. \]  

2.2 Sato’s Theory

Let \( W \) be a pseudo-differential operator,

\[ W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + w_3 \partial^{-3} + \cdots, \]  

where \( w_j \) (j=1,2, ..., m, ...) are functions of \( x \) and \( \partial^{-n} \) is defined by

\[ \partial^{-n} = \left( \frac{\partial}{\partial x} \right)^{-n}. \]  

The inverse operator \( W^{-1} \) exists and can be written as

\[ W^{-1} = v_0 + v_1 \partial^{-1} + v_2 \partial^{-2} + v_3 \partial^{-3} + \cdots, \]  

where

\[ v_0 = 1, \]  

\[ v_1 = -w_1, \]  

\[ v_2 = -w_2 + w_1^2, \]  

\[ v_3 = -w_3 + 2w_1 w_2 - w_1 w_1^2 - w_2^2, \]  

We introduce the term \( H(x; t) \) as

\[ H(x; t) = \begin{pmatrix} h_0^{(0)} & h_0^{(2)} & \cdots \ h_0^{(m)} \\ h_1^{(0)} & h_1^{(2)} & \cdots \ h_1^{(m)} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \]  

where equation (2.13) is the denominator of the function \( W_m(x; t) \).
Let us consider the partial differential equation which depends on $x$ and $t$ as,
\[
W_m(x; t) \frac{\partial^n h_0^{(j)}(x; t)}{\partial t^n} = (\partial^m + w_1(x; t) \partial^{m-1} + \ldots + w_m(x; t)\partial^0)h_0^{(j)}(x; t) = 0
\]
where $j = 1, 2, ..., m$, and
\[
W_m(x; t) = \frac{h_0^{(1)} \ldots h_0^{(m)}}{h_0^{(1)} \ldots h_0^{(m)}}
\]  
(2.14)

After differentiating (2.14) with respect to $t_n$ and solving the remaining equation, we have
\[
B_n = (W \partial^n W^{-1})^+, \quad (2.15)
\]
where $(\cdot)^+$ denotes the differential part of the operator. Equation (2.15) is called the Sato equation. It is important to note that in the derivation of the Sato equation, we have assumed that the solutions take the form (2.15) is called the Sato equation. It is important to note that in the derivation of the Sato equation, we have assumed that the solutions take the form $\frac{\partial^n h_0^{(j)}(x; t)}{\partial t^n} = 0$ (2.16).

Let us define the $\tau$-function as the determinant of $H(x; t)$, the denominator of the function $W_m(x; t)$,
\[
\tau(x; t) = \frac{1}{\det \left( \begin{array}{cccc}
1 & p_1 & p_2 & \ldots \\
0 & 1 & p_1 & p_2 \\
0 & 0 & 1 & p_1 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right) \begin{array}{c}
P_1 \\
P_2 \\
P_3 \\
\vdots \\
\end{array} \begin{array}{c}
\xi_0^{(1)} \\
\xi_0^{(2)} \\
\xi_0^{(m)} \\
\vdots \\
\end{array} \begin{array}{c}
\xi_1^{(1)} \\
\xi_1^{(2)} \\
\xi_1^{(m)} \\
\vdots \\
\end{array} \}
\]
\[
= \det \left( Z_0^x e^{\eta(t, A)} \right), \quad (2.16)
\]
where $Z_0^x$ is a $m \times \infty$ matrix defined by
\[
Z_0^x = \left( \begin{array}{cccc}
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right) \quad (2.17)
\]

By using the expansion theorem on the determinant of product of matrices, $\tau(t)$ in equation (2.16) can be expanded as a sum of products of determinants,
\[
\tau(t) = \sum_{0 \leq l_i < \ldots < -c_{m \cdot \infty}} \left[ \begin{array}{cccc}
p_{1} & p_{1-1} & \ldots & p_{1-m} \\
p_{1-1} & p_{1-2} & \ldots & p_{1-m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1-m} & p_{1-m} & \ldots & p_{1-m} \\
\end{array} \right] \times \left[ \begin{array}{c}
p_{1} \\
p_{1-1} \xi^{(1)}_1 \\
p_{1-2} \xi^{(2)}_1 \\
\vdots \\
p_{1-m} \xi^{(m)}_1 \\
\end{array} \right] \times \ldots \\
\left[ \begin{array}{c}
p_{1} \\
p_{1-1} \xi^{(1)}_m \\
p_{1-2} \xi^{(2)}_m \\
\vdots \\
p_{1-m} \xi^{(m)}_m \\
\end{array} \right], \quad (2.18)
\]
where the summation comprised of all possible combinations of $m$ nonnegative numbers. It is also known that the determinants, composed of $p_j$’s in equation (2.16), are the Schur functions [5]. We may denote this by
\[
S_Y(t) = \left[ \begin{array}{cccc}
p_{1} & p_{1-1} & \ldots & p_{1-m} \\
p_{1-1} & p_{1-2} & \ldots & p_{1-m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1-m} & p_{1-m} & \ldots & p_{1-m} \\
\end{array} \right] \quad (2.19)
\]
and
\[
\xi_Y = \left[ \begin{array}{cccc}
\xi^{(1)}_1 \\
\xi^{(2)}_1 \\
\vdots \\
\xi^{(m)}_1 \\
\xi^{(1)}_m \\
\xi^{(2)}_m \\
\vdots \\
\xi^{(m)}_m \\
\end{array} \right], \quad (2.20)
\]

where the suffix $Y$ stands for the Young diagram that corresponds to the set of numbers $(\xi_1, \xi_2, \ldots, \xi_m)$. The Young diagram is introduced to classify the irreducible representation of the symmetric group (e.g. [18-19]). It is noted that, although different sets of numbers may correspond to a certain $Y$ if $m$ is not fixed, the RHS of equation (2.19) gives the same function for those sets.

Hence, equation (2.18) can be written as
\[
\tau(t) = \sum_{0 \leq Y \leq m} S_Y(t) \xi_Y, \quad (2.21)
\]
where the summation includes all the Young diagrams which have less than $m + 1$ rows. For the coefficients, $\xi_Y$’s in equation (2.20), there exist constraints that are called the Plucker relations which $\xi_Y$’s must always satisfy. This is defined in the next section.

### 3.0 RESULTS AND DISCUSSIONS

First, let us define the Plucker relations in the form of
\[
\sum_{i=1}^{m+1} (-1)^{i} \xi_{Y_1} \xi_{Y_2} = 0, \quad (3.1)
\]
where $Y_1$ is the Young diagram corresponding to $(k_1, \ldots, k_{\ell_1} l_{\ell_1} k_{\ell_1+1}, \ldots, k_{m-1})$ and $Y_2$ to $(l_1, \ldots, l_{\ell_2} l_{\ell_2+1}, \ldots, l_{m-1})$, and where $k_j < l_j < k_{j+1}$ is satisfied (refer [4], [18-19]). As mentioned before, the coefficients $\xi_Y$’s will always satisfy certain constraints called the Plucker relations. Therefore, for all $k_j$ and $l_{\ell_1}$, equation (3.1) will give an infinite number of constraints on the $\xi_Y$’s which are the Plucker relations.

The following subsections will show how the Plucker relations can be transformed into the Hirota bilinear form of KP, KdV and Sawada-Kotera equations for their respective $\tau$-
functions, and thus demonstrates the above-mentioned conceptual framework of Hirota-Sato formalism.

3.1 KP Equation

We now wish to show that the $\tau$-function which satisfies the KP equation is of the same form as the Plucker relations, thus verifying the framework.

From equation (2.16), the definition of $\tau(t)$, we may write

$$\tau(t + s) = \det \left( \mathcal{E}_0 e^{s \eta(t, \lambda)} \mathcal{E}(s) \right).$$  

(3.3)

We denote that

$$\mathcal{E}(s) = e^{s \eta(s, \lambda)} \mathcal{E},$$  

(3.4)

hence, equation (3.3) can be expressed as

$$\tau(t + s) = \sum \xi_r \eta_r(t) \xi_r(s),$$  

(3.5)

where $\xi_r(s)$ is in the form of equation (2.20) and satisfies all the Plucker relations with the parameters $s = (s_1, s_2, s_3, \ldots)$. Now, applying $\xi_r(\tilde{a}_i)$ to (3.5) and using the following orthogonality condition, we have

$$\xi_r(\tilde{a}_i) \eta(t + s) = \xi_r(\tilde{a}_i) \xi_r(s)$$  

(3.6)

$$\xi_r(\tilde{a}_i) \eta(t + s) = \xi_r(s).$$  

(3.7)

We let $t = 0$,

$$\xi_r(\tilde{a}_i) \eta(t + s)\bigg|_{t=0} = \xi_r(\tilde{a}_i) \xi_r(s)$$  

(3.8)

Substitute (3.8) into (3.1) yields

$$\sum(-1)^i \xi_r(\tilde{a}_i) \eta(t) \frac{\partial}{\partial t_i} \eta(\tilde{a}_i) \eta(t) = 0.$$  

(3.9)

Expanding equation (3.9) we have,

for $m = 3, j = 1, \delta = 3, i = 1$,

$Y_1$ corresponding to $(k_1, l_1, k_2)$ while $Y_2$ corresponding to $(l_2, l_3, l_4)$.

For $m = 3, j = 1, \delta = 4, i = 2$,

$Y_1$ corresponding to $(k_1, l_1, k_2)$ while $Y_2$ corresponding to $(l_1, l_3, l_4)$.

For $m = 3, j = 1, \delta = 5, i = 3$,

$Y_1$ corresponding to $(k_1, l_1, k_2)$ while $Y_2$ corresponding to $(l_1, l_2, l_3)$.

For $m = 3, j = 1, \delta = 6, i = 4$,

$Y_1$ corresponding to $(k_1, l_1, k_2)$ while $Y_2$ corresponding to $(l_1, l_2, l_3)$.

Therefore, the corresponding Young diagram of $Y_1$ and $Y_2$ of equation (3.9) yields,

$$-(k_1 t_1 + k_2 t_4 + l_2 t_3 - l_3 t_4) + (k_1 t_1 + k_2 t_4 + l_1 t_3 - l_3 t_4) - (k_1 t_1 + k_2 t_2 + l_1 t_4 - l_4 t_3) = 0$$  

(3.10)

We let $(k_1, k_2, l_1, l_2, l_3, l_4) = (0, 1, 2, 3, 4, 0),$

$$-(0 1 2)(0 3 4) + (0 1 3)(0 2 4) - (0 1 4)(0 2 3) + (0 1 0)(2 3 4) = 0$$  

(3.11)

where we have construed equation (3.11) in the fashion of equation (3.9), and thus we have the Young diagram in the form of

$$S_0(\tilde{a}_i) \tau(t) S(\tilde{a}_i) \tau(t) - S(\tilde{a}_i) \tau(t) S(\tilde{a}_i) \tau(t)$$  

(3.12)

where

$$\tilde{a}_i = \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}, \ldots \right).$$

By using the definition of $S_0(t)$ in (2.19), we obtain the value of

$$S_0 = 1, S = t_1 S = \frac{t_1^2}{2} + t_2, S = \frac{t_1^2}{2} - t_2, S = \frac{t_1^2}{3} - t_3$$

and

$$S = \frac{t_1^2}{12} - t_1 t_3 + t_2^2.$$  

Substituting the value of $S_0$ into (3.12), yields

$$\tau(t) \left( \frac{1}{12} \frac{\partial^4}{\partial t_1^4} + \frac{11}{2} \frac{\partial^2}{\partial t_1^3} \frac{\partial^2}{\partial t_2^2} - \frac{1}{3} \frac{\partial^3}{\partial t_1^2} \frac{\partial}{\partial t_3} \right) \tau(t)$$

$$- \frac{\partial}{\partial t_1} \left( \frac{1}{3} \frac{\partial^3}{\partial t_1^2} - \frac{\partial}{\partial t_3} \right) \tau(t)$$

$$+ \left( \frac{1}{2} \frac{\partial^2}{\partial t_1^2} + \frac{1}{2} \frac{\partial}{\partial t_1} \right) \tau(t) \left( \frac{1}{2} \frac{\partial^2}{\partial t_1^2} + \frac{1}{2} \frac{\partial}{\partial t_1} \right) \tau(t) = 0.$$  

(3.13)

If we simplify further the above equation, then we have

$$\tau(t) \left( \frac{1}{12} \frac{\partial^4}{\partial t_1^4} + \frac{1}{4} \frac{\partial^2}{\partial t_1^2} \frac{\partial^2}{\partial t_2^2} - \frac{1}{3} \frac{\partial^3}{\partial t_1^2} \frac{\partial}{\partial t_3} \right) \tau(t)$$

$$- \frac{\partial}{\partial t_1} \left( \frac{1}{3} \frac{\partial^3}{\partial t_1^2} - \frac{\partial}{\partial t_3} \right) \tau(t)$$

$$+ \left( \frac{1}{2} \frac{\partial^2}{\partial t_1^2} + \frac{1}{2} \frac{\partial}{\partial t_1} \right) \tau(t) \left( \frac{1}{2} \frac{\partial^2}{\partial t_1^2} + \frac{1}{2} \frac{\partial}{\partial t_1} \right) \tau(t) = 0.$$  

(3.14)

We multiply (3.13) by 24 (so as to obtain the standard form of the KP equation), hence we have

$$24 \tau(t) + 6 \tau(t)$$

(3.14)

By applying the $D$-operator properties, we then obtained the KP equation as
\((4D_{t_1}D_{t_3} - D_{t_1}^2 - 3D_{t_3}^2) \tau \cdot \tau = 0.\)  

(3.15)

### 3.2 KdV Equation

In obtaining the bilinear form of KdV equation, we consider only \(t_1\) and \(t_3\) in the formulation of KP equation, since KdV equation is a one dimensional equation of KP. By substituting the value of \(S_y\) into (3.12), this yields

\[
\tau(t) \left( \frac{1}{12} \frac{d^6}{dt^6} - \frac{1}{6} \frac{d^3}{dt^3} \right) \tau(t) - \frac{d}{dt} \tau(t) + \frac{1}{2} \left( \frac{d^3}{dt^3} - \frac{d}{dt} \right) \tau(t) = 0.
\]

(3.16)

We multiply (3.16) by 24, then we have

\[
2\tau(t) - 8\tau(t,t_1,t_3) - 8\tau(t,t_1,t_3) + 8\tau(t,t_1,t_3) + 6\tau_{t_1}^2 = 0.
\]

(3.17)

Rearranging the above equation, we obtain

\[
2\tau(t) - 8\tau(t,t_1,t_3) + 6\tau_{t_1}^2 + 4\left(2\tau(t,t_1,t_3) - 2\tau(t,t_1)\right) = 0.
\]

(3.18)

Expressing in the D-operator terms, we then obtain KdV equation

\[
\left(D_{t_1}^4 - 4D_{t_1}D_{t_3}\right) \tau \cdot \tau = 0.
\]

(3.19)

### 3.3 Sawada-Kotera Equation

In obtaining the bilinear form of Sawada-Kotera equation we consider only \(t_1\) and \(t_3\), since the Sawada-Kotera equation is in the fifth order form of KdV equation.

First, we let \((k_1,k_2,l_1,l_2,l_3,l_4) = (0,1,2,3,1,5)\), and substitute this into equation (3.10),

\[
-(012)(135) + (013)(125) - (014)(124) + (015)(123) - (011)(235) = 0
\]

(3.20)

where if we interpret equation (3.20) in the structure of equation (3.9), then we obtain the form of the Young diagram as

\[
S_y\left(\hat{\nu}_1\right) \tau(t) = \left(\hat{\nu}_1\right) \tau(t) - S\left(\hat{\nu}_1\right) \tau(t) S\left(\hat{\nu}_1\right) \tau(t) = 0.
\]

(3.21)

By using the definition of \(S_y\) in (2.19), we then obtain the following results

\[
S_y = 1, \quad S_y = t_{t_1}, \quad S = \frac{t_1^2}{6}, \quad S = \frac{t_3}{6}, \quad S = t_3 + t_5, \quad S = \frac{t_1}{2} + t_3, \quad S = \frac{t_1^2}{4} + t_3 + t_5, \quad S = \frac{t_4}{6}.
\]

Substituting the value of \(S_y\) into (3.21), yields

\[
2\tau(t) \left( \frac{d^6}{dt^6} + \frac{9}{5} \frac{d^3}{dt^3} \frac{d}{dt} \right) \tau(t) - \frac{d}{dt} \tau(t) + \frac{12}{5} \tau(t) - \frac{18}{5} \tau(t) = 0.
\]

(3.22)

Equivalently, we have

\[
2\tau(t) - 12\tau(t,t_1,t_3) + 36\tau(t,t_1,t_3) - 20\tau(t,t_1,t_3) + 18\left(\tau(t,t_1,t_3) - \tau(t,t_1)\right) = 0.
\]

(3.23)

By applying the D-operator properties, we then obtain the Sawada-Kotera equation

\[
\left(D_{t_1}^6 + D_{t_1}D_{t_3}\right) \tau \cdot \tau = 0.
\]

(3.24)

### 4.0 CONCLUSION

The \(\tau\)-function has acted as a key function to express the solutions of the Sato equation and this is generated from Sato theory. From the nonlinear waves equation being considered, i.e. the KP, KdV and Sawada-Kotera equations, it is shown that the Plucker relations can be represented by the coefficients of \(\tau\)-function. We then deduce that the \(\tau\)-function in the bilinear forms of Hirota scheme are closely associated to the Plucker relations in Sato theory. Therefore, we may conclude that the \(\tau\)-function is essential in indicating this relation between Hirota’s direct method and the Plucker relations, and Sato equation in Sato’s theory. The above deliberations showed that Hirota’s method is linked to Sato theory, and the Hirota-Sato formalism brings to light that the \(\tau\)-function, which underlies the analytic form of soliton solutions of the related nonlinear waves equations does act as the important function to express the solutions of Sato equation. The Kadomtsev-Petvishvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera equations have been used to corroborate this theoretical framework.

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### References


