IMPROVEMENT ON ESTIMATING MEDIAN FOR FINITE POPULATION USING AUXILIARY VARIABLES IN DOUBLE-SAMPLING

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Abstract

The aim of the study is to suggest a difference-cum-ratio type of median estimator for finite population median using two-auxiliary variables in double sampling. Using simple random sampling without-replacement scheme (SRSWOR) the estimated mean square error (MSE) and BIAS are computed for the new suggested median estimator. The suggested median estimator has a smaller MSE than all other median estimators currently in practice, showing a valid contribution to the literature. In addition, some members of the suggested estimator and theoretical comparison of MSE are also computed. Finally, the numerical and graphical comparison of percent relative efficiency (PRE) is also computed for five different real data sets. The PRE of the suggested estimator were 13610.69\%, 177.59\%, 17626.95\%, 204.13\% and 181.29\% for dataset I, II, III, IV and V respectively which reveals the importance of the new estimator.

Keywords: Auxiliary variables, BIAS, median, mean square error, percentage relative efficiency

1.0 INTRODUCTION

In this study a new estimator is suggested for finite population median in double sampling using information of two-auxiliary variables. In real life most of the data are non-normal or in other words highly skewed. Thus, in case of highly skewed distribution in survey sampling which is more frequent the preferred average is median because it handle the outliers in the data sets. Considerable amount of literature e.g Gross [1], Kuk and Mak [2], Allen et al., [3], Singh et al., ([4], [5]), Gupta et al., [6] have studied the estimation of
median in double sampling. Most recently the seminal work of Gupta et al., [6] has suggested a new median estimator in double sampling having the MSE equal to the estimator suggested by Singh et al., [5], the only improvement was in the BIAS factor. However, the study of Gupta et al., [6] provided less BIAS median estimator as compared to the Singh et al., [5] but not more efficient. Therefore, in this study we suggested a new median estimator which is more efficient.

Let us consider a finite population \( U_i = (1, 2, 3, ..., j, ..., N_1) \). Suppose that study variable is denoted by \( A \), the first auxiliary variable denoted by \( B \) and the second auxiliary variable denoted by \( C \). The sample values of the respective variables are denoted by \( a_j, b_j \) and \( c_j \) where \( j = 1, 2, 3, ..., N_1 \) selected by the method of SRSWOR method from the known finite population. Our supposition is that the variables \( A \) and \( B \) are strongly connected with each other but no information on population median \( M_a \) is available, and information on the second auxiliary-variable \( C \) on all units of the population is available (closely connected to the first auxiliary variable \( B \) but slightly connected to the study variable \( A \)). The double sampling or two phase sampling scheme are as follows:

- The sample in first phase i.e., of size \( n_1 \) \( (s_1 \subset U_1) \) is selected to observe only \( B \) for obtaining an estimate of \( M_B \).
- From the sample \( s_1 \) of size \( n_1 \) on first phase, the second phase sample of size \( n_2 \) \( (s_2 \subset s_1) \) is selected to observe \( A, B, \) and \( C \).

Let us suppose that \( M_a, M_b, \) and \( M_c \) are the respective subscripts population medians and the sample medians are denoted by \( \hat{M}_a, \hat{M}_b, \) and \( \hat{M}_c \). Let the second phase sample medians are denoted by \( \hat{M}_a, \hat{M}_b, \) and \( \hat{M}_c \), while the first phase sample medians are denoted by \( \tilde{M}_a, \tilde{M}_b, \) and \( \tilde{M}_c \). The correlation coefficient between sampling distribution of \( \hat{M}_a \) and \( \tilde{M}_a \) are denoted by \( \rho_{ab} \) which is defined as \( \rho_{ab} = \rho(\tilde{M}_a, \tilde{M}_b) = \frac{4(P_{11}(a,b) - 1)}{\rho_{bc}} \) where \( P_{11}(a,b) = P(B \leq M_B \land A \leq M_A) \). The bivariate variables \((A, B)\) distribution tends to a continuous distribution (when \( N_1 \to \infty \)) with their respective marginal densities for \( A \) and \( B \). The same, we describe \( \rho_{bc} \) as \( \rho_{bc} = \rho(\tilde{M}_b, \tilde{M}_c) = 4(P_{11}(b,c) - 1) \), where \( P_{11}(b,c) = P(B \leq M_B \land C \leq M_C) \) and \( \rho_{ac} = \rho(\tilde{M}_a, \tilde{M}_c) = 4(P_{11}(a,c) - 1) \), where \( P_{11}(a,c) = P(A \leq M_A \land C \leq M_C) \) respectively. Suppose that \( a_{1}, a_{2}, ..., a_{n_1} \) are the values in sample \( a \) in ascending or descending order of magnitude. Let \( t_1 \) be an integer value such that \( A_{(t_1)} \leq M_a \leq A_{(t_1+1)} \) and let \( P_1 = \frac{1}{n_1} \) is the subset of \( A \) and the sample values that are less than or equal to the value \( M_a \) (Singh et al., [7]).

2.0 METHODOLOGY

In this section the methodology used for the new suggested median estimator and previous estimators suggested by various authors will be discuss in details.

2.1 Various Estimators of Median Suggested by Different Authors

First of all the following median estimators are discuss and later will be used for comparison before introducing the new suggested estimator. The suggested estimators by numerous authors and their variances, mean squares errors and BIAS expressions are as follows:

[a] Median estimator (per unit)

Gross, S.T. [1] proposed the given estimator:

\[
\hat{M}_{Gross} = \hat{M}_a
\]

The mean square error of \( \hat{M}_{Gross} \) is as follows:

\[
MSE(\hat{M}_{Gross}) = \frac{1}{4f_a(M_a)^2}
\]

(b) Median estimator (Difference type)

Median estimator (Difference type) given as follows:

\[
\hat{M}_{Diff} = \hat{M}_a + y_1(\hat{M}_a - \hat{M}_b),
\]

where \( y_1 \) is constant. The minimum MSE of \( \hat{M}_{Difference} \) are given as follows:

\[
MSE(\hat{M}_Diff)_{min} = \frac{1}{4f_a(M_a)^2}(\frac{1}{m_1} - \frac{1}{n_1}) - \left(\frac{1}{m_1} - \frac{1}{n_1}\right)\rho_{ab}^2
\]

(c) Median Estimator (Ratio)

Singh, S. et al., [4] proposed the ratio estimator for median are as follows:

\[
\hat{M}_{Sing} = \frac{\hat{M}_a}{\hat{M}_b}
\]

The BIAS and MSE of the estimator \( \hat{M}_{Sing} \) are as follows:

\[
Bias(\hat{M}_{Sing}) \geq \frac{M_a}{4(M_aM_b)^2}(\frac{1}{m_1} - \frac{1}{n_1})[1 - \rho_{ab}(M_aM_b)]
\]

and

\[
MSE(\hat{M}_{Sing}) \geq \frac{1}{4f_a(M_a)^2}(\frac{1}{m_1} - \frac{1}{n_1}N_1) + \left(\frac{1}{m_1} - \frac{1}{n_1}\right)^2(M_aM_b)
\]

(d) Median Estimator (ratio type)

Following Srivastava, S.K. [8], a ratio type median estimator are as follows:

\[
\hat{M}_{Sriv} = \hat{M}_a(\frac{\hat{M}_a}{M_a})\delta_1
\]

where \( \delta_1 \) is constant. The minimum BIAS and MSE of \( \hat{M}_{Sriv} \) are as follows:

\[
Bias(\hat{M}_{Sriv})_{min} \geq \frac{\rho_{ab}}{8M_a(M_aM_b)^2}(\frac{1}{m_1} - \frac{1}{n_1})\left[1 - \rho_{ab}(M_aM_b)\right]
\]

and
\[ \text{MSE}(\hat{M}_{\text{Singh}})_{\text{min}} \equiv \frac{1}{4f_1(M_a^2)} \left\{ \left( \frac{1}{m_1} - \frac{1}{n_1} \right) \rho_{ab}^2 \right\} \] (10)

\[
\delta_{1,\text{opt}} = \rho_{ab} \frac{M_{afb}(M_a)}{M_{afa}(M_a)}
\]

(e) Median estimator (Chain ratio type)
Following Chand, L. [9], a median estimator (chain ratio type) is as follows:
\[
\hat{M}_{\text{Chand}} = \hat{M}_a \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right)
\]
(11)
The BIAS and MSE expression respectively of \( \hat{M}_{\text{Chand}} \) are follows:
\[
\text{Bias}(\hat{M}_{\text{Chand}}) \equiv \frac{M_a}{4f_1(M_a^2)} \left\{ 1 - \frac{1}{n_1} \left( \frac{1}{m_1} \right) \rho_{ab} \left( \frac{M_{afb}(M_a)}{M_{afa}(M_a)} \right) \right\}
\]
\[+ \frac{1}{(M_{afa}(M_a))^2} \left\{ 1 - \rho_{ac} \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \right\} \] (12)

and
\[
\text{MSE}(\hat{M}_{\text{Chand}}) \equiv \frac{M_a}{4f_1(M_a^2)} \left\{ 1 - \frac{1}{n_1} \left( \frac{1}{m_1} \right) \rho_{ab} \left( \frac{M_{afb}(M_a)}{M_{afa}(M_a)} \right) \right\}
\]
\[+ \frac{1}{(M_{afa}(M_a))^2} \left\{ 1 - \rho_{ac} \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \right\} \] (13)

(f) Median Estimator (Power-chain-type-ratio)
Srivastava, S.K. et al., [10] median estimator (power-chain-type-ratio) is follows:
\[
\hat{M}_{\text{Srivastava}} = \hat{M}_a \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right)
\]
(14)
where \( \theta_k (k = 1,2) \) are constants. The respective expression of the minimum BIAS and MSE of \( \hat{M}_{\text{Srivastava}} \) are follows:
\[
\text{Bias}(\hat{M}_{\text{Srivastava}})_{\text{min}} \equiv \frac{1}{f_1(M_a^2)} \left\{ \rho_{ab} \right\}
\]
\[\times \left\{ \frac{M_{afb}(M_a)}{M_{afa}(M_a)} \right\} \left\{ 1 - \frac{1}{n_1} \left( \frac{1}{m_1} \right) \rho_{ab} \left( \frac{M_{afb}(M_a)}{M_{afa}(M_a)} \right) \right\} \] (15)
and
\[
\text{MSE}(\hat{M}_{\text{Srivastava}})_{\text{min}} \equiv \frac{1}{4f_1(M_a^2)} \left\{ \left( \frac{1}{m_1} - \frac{1}{n_1} \right) \rho_{ab}^2 \right\}
\]
\[+ \frac{1}{(M_{afa}(M_a))^2} \left\{ \frac{1}{n_1} \left( \frac{1}{m_1} \right) \rho_{ac} \right\} \] (16)
for
\[
\theta_{1,\text{opt}} = \rho_{ab} \left( \frac{M_{afb}(M_a)}{M_{afa}(M_a)} \right) \quad \text{and} \quad \theta_{2,\text{opt}} = \rho_{ac} \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right)
\]
The MSE of the median estimator (power-chain-type-ratio) \( \hat{M}_{\text{Srivastava}} \) in (14) is equivalent to the variance of difference-type median estimator in double sampling using two auxiliary variables are as follows: \( \hat{M}_{\text{Diff}} = \hat{M}_a + t_1(\hat{M}_a - \hat{M}_c) + t_2(\hat{M}_c - \hat{M}_c') \), where \( t_k (k = 1,2) \) are constants.

(g) Median Estimator by Singh (Ratio-type-estimator)
Singh, S. et al., [5] considered the below median estimator (Ratio-type-estimator):
\[
\hat{M}_{\text{Singh}} = \hat{M}_a \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right)
\]
(17)
where \( \eta_i (i = 1,2,3) \) are constants. The respective expression of the minimum BIAS and MSE of \( \hat{M}_c \) are follows:
\[
\text{Bias}(\hat{M}_{\text{Singh}})_{\text{min}} \equiv \frac{M_a}{\hat{M}_c(M_{afa}(M_a))^2} \left\{ 1 - \frac{1}{n_1} \left( \frac{1}{m_1} \right) \right\} \left( \rho_{ab} - \rho_{ac} \right) \]
\[\times \left\{ \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \right\} \left( \rho_{ac} - \rho_{ac} \right) \] (18)
and
\[
\text{MSE}(\hat{M}_{\text{Singh}})_{\text{min}} \equiv \frac{1}{4f_1(M_a^2)} \left\{ \left( \frac{1}{m_1} - \frac{1}{n_1} \right) \rho_{ab}^2 \right\}
\]
\[+ \frac{1}{(M_{afa}(M_a))^2} \left\{ \frac{1}{n_1} \left( \frac{1}{m_1} \right) \rho_{ac} \right\} \] (19)
where \( R_{abc}^2 = \rho_{abc}^2 + \rho_{abc}^2 - \rho_{abc}^2 \) is the multiple correlation coefficient.

The optimum values of \( \eta_i \) are given as follows:
\[
\eta_{1,\text{opt}} = \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right)
\]
\[\times \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \] (15)
\[
\eta_{2,\text{opt}} = \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \] (18)
and
\[
\eta_{3,\text{opt}} = \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \left( \frac{M_{afa}(M_a)}{M_{afa}(M_a)} \right) \] (19)

(h) Gupta Median estimator
Gupta, S. et al., [6], work out on the median estimator suggested by Singh, S. et al., [5] using two auxiliary variables in double sampling and included the range of the second auxiliary variable as a transformation. The new suggested median estimator of Gupta as follows:
\[
\hat{M}_{\text{Gupta}} = \hat{M}_a \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right)
\]
\[\times \left( \frac{\hat{M}_a}{M_a} \right) \left( \frac{\hat{M}_a}{M_a} \right) \] (20)
where \( \xi_j (j = 1,2,3) \) are constants. The respective mathematical expression of the minimum BIAS and MSE of \( \hat{M}_{\text{Gupta}} \) are follows:
Bias(\(\hat{M}_{\text{Gupta}}\))_{\text{min}} \equiv \frac{M^2}{8(M_a f_a(M_a))}\left(1 - \frac{m}{n}\right)^2 \times \left[\frac{1}{m_1} - \frac{1}{n_1}\right]((\rho_{ab} - \rho_{ac})^2 - 2\rho_{ab}(\rho_{ab} - \rho_{ac}p_{bc})(1 - \rho_{bc}^2)) + \left(\frac{M^2 f_a(M_a)}{M_b f_b(M_b)}\right)\rho_{ab} - \rho_{ac}p_{bc}(1 - \rho_{bc}^2) + \left(\frac{M_f f_a(M_a) + M_b f_b(M_b)}{M_a f_a(M_a) + M_b f_b(M_b)}\right)\rho_{ac} - \rho_{bc}p_{bc}(1 - \rho_{bc}^2) + \left(\frac{M_f f_a(M_a) + M_b f_b(M_b)}{M_a f_a(M_a) + M_b f_b(M_b)}\right)\rho_{bc}(1 - \rho_{bc}^2)

The optimum \(\xi\)'s values are follows:
\[\xi_1(\text{opt}) = \eta_1(\text{opt}) = \frac{1}{M_a + M_b}, \quad \xi_2(\text{opt}) = \eta_2(\text{opt}) = \frac{1}{M_a + M_b}, \quad \xi_3(\text{opt}) = \eta_3(\text{opt}) = \frac{1}{M_a + M_b + R_c}\]

The MSE of the median estimator \(\hat{M}_{\text{Gupta}}\) in (20) is equal to Singh et al., [5] median estimator as shown in (17). However, the expressions of the \(x\) terms of the two median estimators \(\hat{M}_{\text{Gupta}}\) and \(\hat{M}_{\text{Singh}}\) are not equal.

(j) Exponential type of Median estimator
An exponential type of median estimator is given as follows Singh, S. [11].
\[\hat{M}_{\text{Exp}} = \left[t_1 \hat{M}_a + t_2 (\hat{M}_b - \hat{M}_b)\right]\exp\left[\left(\frac{\hat{M}_a - \hat{M}_b}{M_a + M_b}\right)\right]
\]

where \(t_k(k = 1, 2)\) are the two constants which is to be determined. The mathematical expression of the \(x\) terms of the MIA and MSE of \(\hat{M}_{\text{Exp}}\) are:

\[\text{Bias}(\hat{M}_{\text{Exp}}) = \left[(1 - \frac{m}{n})\frac{M_a f_a(M_a)}{M_b f_b(M_b)}\left(\frac{N_1 - n_1}{N_1 n_1}\right)\right] + \frac{N_1 - n_1}{32(M_a f_a(M_a))^2\left(\frac{1}{M_a f_a(M_a)}\right)}
\]

and
\[\text{MSE}(\hat{M}_{\text{Exp}})_{\text{min}} \equiv M^2 - \frac{M^2 f_a(M_a)^2}{4M_a f_a(M_a)^2 + M_b f_b(M_b)^2}\]

The two constants optimum values are follows:
\[\xi_1(\text{opt}) = \frac{2Y}{1 + \frac{M_a f_a(M_a)}{M_b f_b(M_b)}}\left(\frac{N_1 - n_1}{N_1 n_1}\right)\left(\frac{1}{M_a f_a(M_a)}\right)\]

and
\[\xi_2(\text{opt}) = \frac{-\rho_{abc} f_b(M_b)}{2Y^2 f_b(M_b)\left(\frac{1}{M_a f_a(M_a)}\right)^2}\]

where
\[Y = \left[1 + \frac{N_1 - n_1}{M_a f_a(M_a)}\left(\frac{1}{M_a f_a(M_a)}\right)\right] - \frac{N_1 - n_1}{2M_a f_a(M_a)}\left(\frac{1}{M_a f_a(M_a)}\right)\]

and
\[X = \frac{1}{4}\left(\frac{N_1 - n_1}{N_1 n_1}\right)\frac{\rho_{abc} f_b(M_b) f_c(M_c)}{M_a f_a(M_a) f_c(M_c)} + \frac{N_1 - n_1}{16M_a f_a(M_a) f_c(M_c)}\left(\frac{1}{M_a f_a(M_a)}\right)^2 - 2\]

(k) Median estimator (proposed by Nursel)

Nursel, K. [12] have suggested a class of estimators for population median are as follows:
\[\hat{M}_{\text{Nursel}} = \pi_1 \hat{M}_a + \pi_2 (\hat{M}_b - \hat{M}_b) + \pi_3 \left(\frac{M_c}{M_c + n(\hat{M}_c - \hat{M}_c)}\right)\]

where \(\pi_j(j = 1, 2, 3)\) are constants will be determined and \(m, n\) are either function of known parameters of the population of second- auxiliary variable such as skewness \(\beta_1(c)\), kurtosis \(\beta_2(c)\), correlation-coefficient \(\rho_{bc}\) etc or constants. Various estimators can be generated by giving suitable values to \(m, n\). The expression of bias and minimum MSE respectively of the estimator \(\hat{M}_{\text{Nursel}}\) are given as follows:

\[\text{Bias}(\hat{M}_{\text{Nursel}}) \equiv (\pi_1 - 1)M_a + \pi_2 M_a + \pi_3 \left(\frac{m(m - 2n)}{2} - m + 1\right)\left(\frac{1}{M_c + n(\hat{M}_c - \hat{M}_c)}\right)\]

and
\[\text{MSE}(\hat{M}_{\text{Nursel}})_{\text{min}} \equiv M^2 - \frac{M^2 f_a(M_a)^2}{4M_a f_a(M_a)^2 + M_b f_b(M_b)^2}\]

where
\[Q = \left[1 + \frac{N_1 - n_1}{M_a f_a(M_a)}\right]^2, \quad R = \left[1 + \frac{2M_a f_a(M_a)}{M_b f_b(M_b)}\right]^2, \quad S = \left[2 + \frac{2M_a f_a(M_a)}{M_b f_b(M_b)}\right]^2, \quad T = \left[1 + \frac{2M_a f_a(M_a)}{M_b f_b(M_b)}\right]^2\]

and
\[\hat{M}_{\text{Aamir}} = k_1 \hat{M}_a + k_2 (\hat{M}_b - \hat{M}_b) + k_3 \exp\left[\frac{q(M_a f_a(M_a))}{(r + q(M_a f_a(M_a)))} + \frac{q(M_a f_a(M_a))}{(r + q(M_a f_a(M_a)))}\right]\]

where \(k_j(j = 1, 2, 3)\) are constants and function of known population parameters of second auxiliary variable such as correlation-coefficient \(\rho_{bc}\), skewness, kurtosis, Range or either constants. The respective minimum expression of \(x\) terms of MIA and MSE of \(\hat{M}_{\text{Aamir}}\) are as follows:

\[\text{Bias}(\hat{M}_{\text{Aamir}}) = \frac{M_a}{2}\left[Z_2 \gamma f_a(M_a) - Z_1 - Z_4 \left(2M_a f_a(M_a)^2 + r^2 \left(\frac{1}{4n} - \frac{1}{4m}\right)\right)\right]\]

and
\[\text{MSE}(\hat{M}_{\text{Aamir}}) = \frac{M^2}{4}\left[Z_2 \gamma f_a(M_a) - Z_1 - Z_4 \left(2M_a f_a(M_a)^2 + r^2 \left(\frac{1}{4n} - \frac{1}{4m}\right)\right)\right] + 2\gamma^2(2Z_2 + 2Z_4) - Z_2^2\left(\frac{1}{4n} + \frac{1}{4m}\right)\]

and
\[Z_4 M_a f_a(M_a)\left(\frac{1 \pm \frac{1}{m} \mp \frac{1}{n}}{m^n \times m^n}\right) + Z_2^2\left(\frac{1}{4n} + \frac{1}{4m}\right)\]
The optimum values of \( k_j (j = 1, 2, 3) \) are as follows:

\[
K_1 = \frac{2\gamma M_a f(M_a)}{\varepsilon_1}, \quad K_2 = \frac{2\gamma M_a f(M_a^2)}{\varepsilon_1}, \quad \text{and} \quad K_3 = \frac{2\gamma M_a f^2(M_a)}{\varepsilon_1}
\]

where

\[
\gamma = \frac{q M_z}{2r + q M_z}
\]

### 2.2 The New Proposed Median Estimator

Motivated by the modified versions of median estimators from Singh, S. et al., [5], Gupta, S. et al., [6], Nurset, K. [12], Aamir, M. et al., [13] and Jhajj and Waila [14], we suggest a generalized difference-cum-ratio type of median estimator in double sampling using two auxiliary variables for the finite population median. The new suggested estimator is as follows:

\[
\hat{M}_{propose} = \left( \frac{\hat{M}_a}{\hat{M}_a} \right) \left( \frac{\hat{M}_a}{\hat{M}_b} \right) \left( \frac{\hat{M}_a}{\hat{M}_c} \right) \left( \frac{\hat{M}_a}{\hat{M}_b} \right) \left( \frac{\hat{M}_a}{\hat{M}_c} \right)
\]

where \( k_j (j = 1, 2, 3) \) and \( \theta \) are constants. The main purpose of using \( \hat{M}_{propose} \) as in (32) is to increase the precision of the median estimator by taking the relevant advantage of the correlation between \( a, b, a, c, \) and \( b \) and \( c \) to calculate the properties of the proposed-estimator \( \hat{M}_{propose} \) to the first-order of approximation. Let

\[
e_{10} = \left( \frac{\hat{M}_a - M_a}{M_a} \right), \quad e_{11} = \left( \frac{\hat{M}_a - M_a}{M_a} \right), \quad e_{12} = \left( \frac{\hat{M}_b - M_b}{M_b} \right), \quad e_{13} = \left( \frac{\hat{M}_b - M_b}{M_b} \right), \quad e_{14} = \left( \frac{\hat{M}_c - M_c}{M_c} \right), \quad e_{15} = \left( \frac{\hat{M}_c - M_c}{M_c} \right)
\]

Substituting the values of \( e_{11}'s \) in (32), and we also assume that \( |e_{11}| < 1 \), \( l = 10, 11, 12, 13, 14, 15 \), therefore we expand \( \hat{M}_{propose} \), by using second-degree of approximation, we have

\[
\hat{M}_{propose} = M_a\left[ (1 + e_{10}) + \theta (e_{11} - e_{10}) \right] \left[ 1 + \kappa_1(1 - \theta)(e_{12} - e_{10}) \right] \left[ 1 + \kappa_2(1 - \theta)(e_{13} - e_{10}) \right] \left[ 1 + \kappa_3(1 - \theta)(e_{14} - e_{10}) \right] \left[ 1 + \kappa_4(1 - \theta)(e_{15} - e_{10}) \right]
\]

Substituting the values of \( e_{11}'s \) from (33), and also using the values of \( e_1, \theta, \kappa_1, \kappa_2, \kappa_3, \) and \( \kappa_4 \), the following mathematical expression can be easily obtained from Sukhatme, S. et al., [15] and Dorfman, A.H. [16].

\[
E(e_{10}^2) = f_{12}K_{a}^2, \quad E(e_{11}^2) = f_{12}K_{a}^2, \quad E(e_{12}^2) = f_{12}K_{a}^2, \quad E(e_{13}^2) = f_{12}K_{a}^2, \quad E(e_{14}^2) = f_{12}K_{a}^2, \quad E(e_{15}^2) = f_{12}K_{a}^2,
\]

Substituting the values of (36), (37), (38) and replacing the values of \( f_{11}, f_{12}, K_{a}, K_{b}, \) and \( K_{c} \) in (34) and (35) and also simplify the expression, the optimum \( BIAS \) and \( MSE \) of the suggested proposed estimator \( \hat{M}_{propose} \) after simplification are given as follows:

\[
\kappa_1(\text{opt}) = \frac{K_{a}}{(1 - p_{ab})K_{b}}, \quad \kappa_2(\text{opt}) = \frac{K_{a}}{(1 - p_{bc})K_{c}}, \quad \kappa_3(\text{opt}) = \frac{K_{b}}{K_{a}}
\]
Bias \( (\hat{M}_{\text{Propose}})_{\min} \) for \( \theta = 0 \):

\[
(1 - \theta)^2 \times \left( \frac{(\rho_{bc} - \rho_{ac})}{2M_{fa}(M_a以外1)} + \frac{1}{M_a} \right) \left( \frac{\theta(1 - \theta)}{2M_{fa}(M_a以外1)} + \frac{1}{M_a} \right)
\]

(39)

and

\[
MSE(\hat{M}_{\text{Propose}})_{\min(\theta=0)} \equiv \frac{1}{4f_a(M_a以外1)} \left[ \frac{1}{m_1} - \frac{1}{n_1} \right] + \left( \frac{1}{m_1} - \frac{1}{n_1} \right) \theta(\theta - 2)
\]

(40)

Thus the equations (39) and (40) represents the minimum bias and mean square error of the proposed estimator \( \hat{M}_{\text{Propose}} \) and the equation (41) represent the minimum mean square error at the point \( \theta = 1 \).

2.3 Some Members of the Suggested General Class of Median Estimators

In this section we compare the proposed median estimator with the other existing median estimators.

(a) Put \( k_1 = 0, k_2 = 0, k_3 = 0 \) and \( \theta = 0 \) in (32), we get the following Gross, S.T. [1] estimator:

\[
\hat{M}_{\text{Gross}} = \hat{M}_a
\]

(42)

(b) Put \( k_1 = 0, k_2 = 0, k_3 = 0 \) and \( \theta \) is constant in (32), we get the following difference type of estimator:

\[
\hat{M}_{\text{Diff}} = \hat{M}_a + \theta (M_b - M_a)
\]

(43)

where \( \theta \) is constant.

(c) Put \( k_1 = 1, k_2 = 0, k_3 = 0 \) and \( \theta = 0 \) in (32), we get the following Singh, S et al., [4] ratio-type median estimator in double sampling:

\[
\hat{M}_{\text{Singh}} = \hat{M}_a \cdot \hat{M}_b
\]

(44)

(iv) Put \( k_1 = 1, k_2 = 0, k_3 = 0 \) and \( \theta = 0 \) in (32), we get the following Srivastava, S.K. [7] median ratio type estimator:

\[
\hat{M}_{\text{Sriv}} = \hat{M}_a \left( \frac{M_{fa}(M_aM_a以外1) - 2\rho_{ab}}{M_{fa}(M_aM_a以外1)} \right) + \left( \frac{1}{n_1} - \frac{1}{n_1} \right) + \frac{1}{n_1} \frac{1}{n_1} \rho_{ac} \geq 0
\]

The above relationship is always true.

(d) Efficiency Condition of the Median Estimator

The proposed median estimator \( \hat{M}_{\text{Propose}} \) is more efficient than the current median estimators if the following conditions are satisfied.

(a) Efficiency Condition (I)

By [2] and (41)

\[
MSE(\hat{M}_{\text{Propose}})_{\min(\theta=1)} \leq MSE(\hat{M}_{\text{Gross}})
\]

if

\[
\rho_{ac} \geq 0
\]

(48)

The above relationship is always true.

(b) Efficiency Condition (II)

By [4] and (41)

\[
MSE(\hat{M}_{\text{Propose}})_{\min(\theta=1)} \leq MSE(\hat{M}_{\text{Diff}})_{\min} if
\]

\[
\left( \frac{1}{m_1} - \frac{1}{n_1} \right) (1 - \rho_{ab}^2) + \left( \frac{1}{n_1} - \frac{1}{n_1} \right) \rho_{ac}^2 \geq 0
\]

(49)

The above relationship is always true.

(c) Efficiency Condition (III)

By [7] and (41)

\[
MSE(\hat{M}_{\text{Propose}})_{\min(\theta=1)} \leq MSE(\hat{M}_{\text{Singh}}) if
\]

\[
\left( \frac{1}{m_1} - \frac{1}{n_1} \right) \left( \frac{M_{fa}(M_aM_a以外1) - 2\rho_{ab}}{M_{fa}(M_aM_a以外1)} \right) + \left( \frac{1}{n_1} - \frac{1}{n_1} \right) + \left( \frac{1}{n_1} \right) \rho_{ac} \geq 0
\]

(50)

The above relationship is always true.

(d) Efficiency Condition (IV)

By (13) and (41)
\[ MSE(\bar{\mu}_{Propose})_{min}\theta=1 \leq MSE(\bar{\mu}_{Chand}) \text{ if } \]
\[ \left( \frac{1}{m_1} - \frac{1}{n_1} \right) (M_{afA}(M_A) - M_{afB}(M_B)) - 2 \rho_{ab} \]
\[ + \left( \frac{1}{m_2} - \frac{1}{n_2} \right) (M_{af\hat{A}}(M_{\hat{A}}) - M_{af\hat{B}}(M_{\hat{B}})) - 2 \rho_{ac} + \left( \frac{1}{m} - \frac{1}{n} \right) (1 - \rho_{ac}) \geq 0. \] (51)

The above relationship is always true.

(e) Efficiency Condition (V)

By (16) and (41)

\[ MSE(\bar{\mu}_{Propose})_{min}\theta=1 \leq MSE(\bar{\mu}_{Singh})_{min} \text{ if } \]
\[ 1 - \rho_{ac} \geq 0. \] (52)

The above relationship is always true.

(f) Efficiency Condition (VI)

By (19) and (41)

\[ MSE(\bar{\mu}_{Propose})_{min}\theta=1 \leq MSE(\bar{\mu}_{Singh})_{min} \text{ if } \]
\[ R_{abc}^2 \leq 1 \] (53)

The above relationship is always true.

(g) Efficiency Condition (VII)

By (25) and (41)

\[ MSE(\bar{\mu}_{Propose})_{min}\theta=1 \leq MSE(\bar{\mu}_{Exp})_{min} \text{ if } \]
\[ M^2_a(Y - \frac{x^2}{4}) + \frac{M^2_{ab} \rho_{oa}}{4f_{m}(M_A)} \left( \frac{m_1 - n_1}{m_1 n_1} \right) \]
\[ - \left( M_{afA}(M_A) \right)^2 + \rho_{ab} \left( \frac{m_1 - n_1}{m_1 n_1} \right) \left( 1 - \frac{1}{n_1} \right) (1 - \rho_{ac}) \geq 0. \] (54)

The above relationship is always true if \( Y > \frac{x^2}{4} \).

We observed from equations (48) to (54), that the proposed-median estimator \( \bar{\mu}_{Propose} \) is better than all other median estimators, i.e., \( \bar{\mu}_e \) \( = \) Gross Difference, Singh, Srivastava, and Singh1, Gupta, Exp, Aamir) for all types of finite populations. In our empirical study we use the different values of \( \theta \) to check the relative performance of the suggested median estimator over the usual median estimators.

3.2 Description of the Numerical Data Sets

For numerical comparison the following sets of data are being used.

(a) Numerical Data set (I) [Source: Agriculture Department of United States [17]]

A: The value of agricultural production in 2009 in million dollars

B: The value of agricultural production of U.S. in 2008 in millions of dollars

C: The value of agricultural production of U.S. in 2007 in millions of dollars

(b) Numerical Data set (II) [Source: Pakistan Ministry of Food and Agriculture [18]]

A: In 2003 tomato production district wise (in tonnes)

B: In 2002 tomato production district wise (in tonnes)

C: In 2001 tomato production district wise (in tonnes)

(c) Numerical Data set (III) [Source: Agriculture Department of United States [17]]

A: The Soybeans production in 2010 (in million bushels)

B: The Soybeans production in 2009 (in million bushels)

C: The Soybeans production in 2008 (in million bushels)

(d) Numerical Data set (IV) [Source: Horticulture Department India [19]]

A: In India State wise major production of spices (in tonnes) 2010-11

B: In India State wise major production of spices (in tonnes) 2009-10

C: In India State wise major production of spices (in tonnes) 2008-09

(e) Numerical Data set (V) [Source: The data taken from Singh [20]]

A: In 1995 the marine recreational fisherman caught the number of fish

B: In 1994 the marine recreational fisherman caught the number of fish

C: In 1993 the marine recreational fisherman caught the number of fish

The descriptive statistics of all five data sets are shown in Table 1 the above data sets also used by (Aamir et al., [21]).

| Table 1 Descriptive statistics from all of the Five data (D) sets |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| D.Set I | D.Set II | D.Set III | D.Set IV | D.Set V |
| N1 | 50.0 | 97.0 | 31.0 | 29.0 | 69.0 |
| n1 | 30.0 | 46.0 | 15.0 | 15.0 | 24.0 |
| m1 | 20.0 | 33.0 | 10.0 | 10.0 | 17.0 |
| Mean(A) | 6617.60 | 3134.61 | 107397.10 | 184.51 | 4513.91 |
| Mean(B) | 7345.49 | 3051.29 | 108354.20 | 139.01 | 4504.99 |
| Mean(C) | 6539.96 | 2744.05 | 95708.89 | 143.01 | 4591.04 |
| \( \rho_{ab} \) | 0.9988 | 0.2010 | 0.9939 | 0.9989 | 0.1560 |
| \( \rho_{bc} \) | 0.9917 | 0.1229 | 0.9950 | 0.0530 | 0.3170 |
| Mean(M2) | 5014.60 | 1241.99 | 43711.00 | 71.42 | 2067.91 |
| Median(M2) | 5652.27 | 1232.90 | 64800.00 | 43.29 | 2010.95 |
| Median(M3) | 5023.68 | 1207.00 | 5014 | 41.70 | 2096.97 |
| \( f_{m}(M_A) \) | 0.000081 | 0.000221 | 0.000011 | 0.000367 | 0.000141 |
| \( f_{b}(M_B) \) | 0.000070 | 0.000022 | 0.000011 | 0.000527 | 0.000141 |
| \( f_{c}(M_C) \) | 0.000081 | 0.000021 | 0.000011 | 0.000321 | 0.000131 |
| Range(A) | 41035.0 | 26406.0 | 485249.0 | 1160.00 | 37975.0 |
| Range(B) | 38705.0 | 61547.0 | 37753.0 | 1224.00 | 34025.0 |
3.3 Percentage Relative Efficiencies of Various Median Estimators

The percentage relative efficiencies are computed as compared to the usual median Gross, S.T. [1] estimator by using the mean square errors of all median estimators for all five numerical data sets as given as follows:

\[
PRE = \frac{MSE(\hat{M}_\text{Gross})}{MSE(\hat{M}_i)} \times 100
\]

Where \( \xi = \text{Gross, Diff, Singh, Sri, Chand, Sri1, Singh1, Gupta, Exp, Aamir, Propose}(\theta_i) \)

and \( \theta_i = 0.10, 0.50, 1.00, 1.50, 1.90 \)

Table 2 contains the \( PRE \)'s of all median estimators over the usual Gross, S.T. [1], median estimator i.e. \( \hat{M}_\text{Gross} \). Here we discussed that which median estimator is more efficient than the other median estimator in double sampling using two auxiliary variables. Thus from the Table 2, we observed that the proposed median estimator i.e. \( \hat{M}_\text{Propose} \) are more efficient than all other median estimators in two-phase sampling scheme using auxiliary information for all values of \( \theta_i \).

\( \text{i.e. } \hat{M}_\xi(= \text{Gross, Diff, Singh, Sri, Chand, Sri1, Singh1, Gupta, Exp, Aamir}). \)

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Percentage relative efficiencies of all data [D] sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimator</td>
<td>D.Set I</td>
</tr>
<tr>
<td>( \hat{M}_\text{Gross} )</td>
<td>100.00</td>
</tr>
<tr>
<td>( \hat{M}_\text{Diff} )</td>
<td>224.30</td>
</tr>
<tr>
<td>( \hat{M}_\text{Singh} )</td>
<td>63.90</td>
</tr>
<tr>
<td>( \hat{M}_\text{Sriv} )</td>
<td>224.30</td>
</tr>
<tr>
<td>( \hat{M}_\text{Gross} )</td>
<td>12.49</td>
</tr>
<tr>
<td>( \hat{M}_\text{Sriv1} )</td>
<td>11375.69</td>
</tr>
<tr>
<td>( \hat{M}_\text{Singh1} )</td>
<td>11429.00</td>
</tr>
<tr>
<td>( \hat{M}_\text{Gupta} )</td>
<td>11429.00</td>
</tr>
<tr>
<td>( \hat{M}_\text{Exp} )</td>
<td>859.03</td>
</tr>
<tr>
<td>( \hat{M}_\text{Aamir} )</td>
<td>11728.38</td>
</tr>
<tr>
<td>( \hat{M}_\text{Propose}(0.1) )</td>
<td>11178.97</td>
</tr>
<tr>
<td>( \hat{M}_\text{Propose}(0.5) )</td>
<td>12990.59</td>
</tr>
<tr>
<td>( \hat{M}_\text{Propose}(1.0) )</td>
<td>13610.69</td>
</tr>
<tr>
<td>( \hat{M}_\text{Propose}(1.5) )</td>
<td>12990.59</td>
</tr>
<tr>
<td>( \hat{M}_\text{Propose}(1.9) )</td>
<td>11787.90</td>
</tr>
</tbody>
</table>

3.4 Graphical Representation of All Median Estimators with respect to Percentage Relative Efficiency

In this section, PRE are presented graphically for all data sets from Figure 1 to Figure 5 respectively. From the graphical representation of percentage relative efficiency it is clear that the suggested median estimator is more efficient than all other existing median estimators for all five different data sets. From the graphical representation of \( PRE \) clearly stated that the suggested median estimator in two-phase sampling scheme using two auxiliary variables is more efficient for the range of values of \( \theta \), i.e. \( 0.10 \leq \theta \leq 1.90 \) and more efficient at the point \( \theta = 1.00 \).
4.0 CONCLUSION

In this study we proved mathematically, empirically as well as graphically that our new suggested estimator $\hat{\theta}_{\text{Propose}}$ is more efficient than all other existing median estimators. The MSE of the suggested estimator is smaller than all other existing median estimators for all data sets. For checking the relative performance of MSE of the suggested estimator, we substitute different values of $\theta$ i.e. (0.10 ≤ $\theta$ ≤ 1.90) to get the minimum MSE for the suggested-estimator and we got the minimum MSE at the point $\theta = 1.00$. Thus we observed that the suggested median estimator produce the minimum MSE at the point $\theta = 1.00$.

Hence, based on this study it is concluded that the suggested median estimator is more efficient than other median estimators in double-sampling. It is recommended that our new suggested estimator for estimating finite population median $\hat{\theta}_{\text{Propose}}$ can be used in practice for obtaining more efficient results.

References