Travelling wave solution to a generalized Kuramoto-Sivashinsky equation

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ABSTRACT

This paper presents a method for the construction of travelling wave solution of the generalized Kuramoto-Sivashinsky equation. The method makes use of a series representation, and is essentially an extension of Hirota's method.

1. INTRODUCTION

The formation of a row of soliton-like pulses in an unstable, dissipative, and dispersive nonlinear system has been observed in the numerical solution of the generalized Kuramoto-Sivashinsky (gKS) equation

\[ u_t + uu_x + au_{xx} + bu_{xxx} + cu_{xxxx} = 0, \]  

where \( a, b \) and \( c \) are positive constants characterizing instability (self-excitation), dispersion, and dissipation, respectively. Equation (1) can be used to describe the long waves on a viscous fluid flowing down an inclined plane [Topper and Kawahara, 1978] and unstable drift waves in plasmas [Cohen et al., 1976].

We remark that, in the limits \( b \to 0 \) or \( c \to 0 \), the gks equation can be approximated by the Kuramoto-Sivashinsky (KS) equation

\[ u_t + uu_x + au_{xx} + cu_{xxxx} = 0, \]  

or the Korteweg-de Vries Burgers' (KdVB) equation

\[ u_t + uu_x + au_{xx} + bu_{xxx} = 0, \]  

respectively.

The KS equation describes chemical reactions which exhibit a turbulent-like behaviour [Wijngaarden, 1972], while the KdVB equation represents the flow of liquids containing gas bubbles [Kuramoto and Tsuzuki, 1976] and the propagation of waves on an elastic tube filled with a viscous fluid [Johnson, 1969]. It is known that the KS equation has travelling wave solutions in terms of hyperbolic tangent function, whereas the KdVB equation has travelling wave solutions involving a combination of hyperbolic secant and tangent functions. A detailed discussion of the KdVB equation has been given by Jeffrey and Mohamad [1991].

2. TRAVELLING WAVE SOLUTION OF THE gKS EQUATION

In this section, travelling wave solution of the generalized gKS equation (1) is found by means of the series method. This method uses an approach similar to that due to Hirota when seeking single and multiple soliton solutions of the KdV equation.
We seek a solution of the gKS equation in the form

$$u(x, t) = \sum_{j=0}^{\infty} u_j F^{j-3}. \quad (4)$$

We express the $x$ and $t$ derivatives of $u(x, t)$ in terms of $u_j(x, t)$ and $F(x, t)$. Inserting the expressions obtained from the derivatives into the gKS equation, we find that

$$u_0 = 120cF_x^3, \quad (5a)$$

$$u_1 = -15bF_x - 180cF_tF_{xx}, \quad (5b)$$

$$u_2 = \frac{15}{76} \left( 16a - \frac{b^2}{c} \right) F_x + 15bF_{xx} = 60cF_{xxx}. \quad (5c)$$

These results are the coefficients of the powers of $F^{-7}$ and $F^{-6}$ and $F^{-5}$. Using equations (5c), the coefficient of $F^{-4}$ can be written in the form

$$F_t - \frac{b}{76c} \left( 7a - \frac{13b^2}{8c} \right) F_x + \frac{15}{152} \left( 16a - \frac{b^2}{c} \right) F_{xx} + 5bF_{xxx} + 15cF_{xxxx}$$

$$- \frac{15}{4} bF_x^2 F_x^{-1} + 15cF_x^2 F_x^{-2} - 30cF_{xx}F_x F_{xx}^{-1} + u_3 F_x = 0. \quad (6)$$

In order to find the coefficients $u_j(x, t)$ for $j \geq 4$, we can use a recursion relation which can be obtained from the condition that the terms multiplying the same powers of $F(x, t)$ are equal. It turns out that the coefficient $u_6$ is not determined by this formula, and it is necessary to set $u_j = 0$ for $j \geq 4$ in equation (4). A solution $u(x, t)$ of the gKS equation (1) may then be written

$$u(x, t) = \frac{u_0}{F^3} = \frac{u_1}{F^2} = \frac{u^2}{F} + u_3(x, t), \quad (7)$$

where $u_0(x, t)$, $u_1(x, t)$ and $u_3(x, t)$ are expressed in terms of derivatives of $F(x, t)$ according to equations (5). Taking into account equations (5), equation (7) can be re-written as

$$u(x, t) = \frac{15}{76} \left( 16a - \frac{b^2}{c} \right) \frac{\partial}{\partial x} \ln F + 15b \frac{\partial}{\partial x^2} \ln F$$

$$+ 60c \frac{\partial^3}{\partial x^3} \ln F + u_3. \quad (8)$$

This equation can be used to find an exact solution of the gKS equation. Now let

$$F(x, t) = 1 + \exp(kx - wt), \quad (9)$$

where $k$ and $\omega$ are constants to be determined. After substitution of (9) into (6) and (8), followed by setting $u_3 = 0$, we find that a function $F$ of the form (9) will be a solution of the gKS equation provided

$$\omega - \frac{15}{152} \left( 16a - \frac{b^2}{c} \right) k^2 = 0 \quad \text{and} \quad \frac{b}{76c} \left( 7a - \frac{13b^2}{8c} \right) k - \frac{5}{4} bk^3 = 0. \quad (10)$$
From (10), we have
\[
\omega = \frac{15}{152} \left( 16a - \frac{b^2}{c} \right) k^2 \text{ and } k^2 = \frac{7a - \frac{13b^2}{8c}}{95c}
\] (11)

Thus the solution of the gKS equation may be written
\[
u(x,t) = \frac{15}{76} \left( 16a - \frac{b^2}{c} \right) \left \{ \frac{1}{2} \right \} \left[ 1 + \tanh \frac{1}{2} (kx - \omega t) \right] \\
+ 15k^2 \text{sech}^2 \frac{1}{2} (kx - \omega t) \left[ \frac{b}{4} - c \tanh \frac{1}{2} (kx - \omega t) \right],
\] (12)
where \( k \) and \( \omega \) are given by (11).

Equation (12) represents a travelling wave solution to the gKS equation. It is clear that, if \( b \to 0 \), the travelling wave solution (12) can be reduced to the travelling wave solution of the KS equation
\[
u(x,t) = \frac{30}{19} a k \left[ 1 + \tanh \frac{1}{2} (kx - \omega t) \right] \\
- 15c k^3 \left[ 1 - \tanh^2 \frac{1}{2} (kx - \omega t) \right] \tanh \frac{1}{2} (kx - \omega t)
\] (13)
where from (11), we have
\[
\omega = \frac{30}{19} a k^2.
\] (14)

This result is in agreement with the result obtained by Kuramoto and Tsuzuki [1976].

Finally, we remark that this method can be generalized to a larger class of nonlinear evolution equations of the form
\[
u_t + auu_x + \sum_{i=2}^{N} a_i u_{ix} = 0
\] (15)

In general, the solution of equation (15) will be of the form
\[
u(x, t) = \sum_{j=0}^{N-1} u_j F^{j-N+1}.
\] (16)

3. ACKNOWLEDGEMENTS

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4. REFERENCES

APPENDIX I: The SCOPE COMMON statements.

SCOPE COMMON
STOCK <- MATERIAL (150, 25)
LCHM1 <- LEFT _ CHAMFER (03, 21, 24, 0, 0, 0, 003, 0, 0)
CYLN1 <- CYLINDER (37, 24, 0, 0, 0, 003, 0, 0)
UCUT1 <- UNDERCUT (10, 13, 0, 0, 0, 040, 0, 0)
CYLN2 <- CYLINDER (20, 18, 0, 0, 0, 050, 0, 0)
RTAP1 <- RIGHT _ TAPER (30, 13, 18, 0, 0, 0070, 0, 0)
CUT2 <- UNDERCUT (05, 10, 0, 0, 0, 100, 0, 0)
CYLN3 <- CYLINDER (25, 13, 0, 0, 0, 105, 0, 0)
CYLN4 <- CYLINDER (18, 08, 0, 0, 0, 130, 0, 0)
RCHMH1 <- RIGHT _ CHAMFER (02, 06, 08, 0, 0, 0, 148, 0, 0)
COMPONENT <- STOCK - LCHM1 - CYLN1 - UCUT1 - CYLN2 - RTAP1 - UCUT2 - CYLN3 - CYLN4 - RCHMH1

! MATERIAL

! SCOPE MATERIAL
SOLID FAMILY MATERIAL (LENGTH,RADIUS):MATERIAL
MATERIAL <- CYL (LENGTH, RADIUS) AT (ROTY =90)
RADIUS = 25

! CYLINDER

! SCOPE CYLINDER
SOLID FAMILY CYLINDER (LENGTH, RADIUS1, X1, Y1, Z1, $X2, Y2, Z2).CYLINDER
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 <- CYL (LENGTH, RADIUS1) AT (ROTY = 90)
CYLINDER <- MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

! RIGHT _ TAPER

! SCOPE RIGHT _ TAPER
SOLID FAMILY RIGHT _ TAPER (LENGTH, RADIUS1, $RADIUS2, X1, Y1, Z1, Y2, Y2, Z2).RIGHT _ TAPER
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 <- CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 90)
RIGHT _ TAPER <- MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

! UNDERCUT

! SCOPE UNDERCUT
SOLID FAMILY UNDERCUT (LENGTH, RADIUS1, X1, Y1, Z1, $X2, Y2, Z2).UNDERCUT
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 < - CYL (LENGTH, RADIUS1) AT (ROTY = 90)
UNDERCUT < - MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
!
! RIGHT _ CHAMFER
!
SCOPE RIGHT _ CHAMFER
SOLID FAMILY RIGHT _ CHAMFER (LENGTH, RADIUS1, RADIUS2, S
X1, Y1, Z1, X2, Y2, Z2) : RIGHT _ CHAMFER
OBJ1 < - CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 < - CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 90)
RIGHT _ CHAMFER < - MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25
!
! LEFT _ CHAMFER
!
SCOPE LEFT _ CHAMFER
SOLID FAMILY LEFT _ CHAMFER (LENGTH, RADIUS1, RADIUS2, S
X1, Y1, Z1, X2, Y2, Z2) : LEFT _ CHAMFER
OBJ1 < - CYL (LENGTH, RADIUS) AT (ROTY = 270)
OBJ2 < - CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 270)
LEFT _ CHAMFER < - MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

SCOPE COMMON
STOCK <- MATERIAL (150, 25)
LCHM1 <- LEFT _CHAMFER (03, 21, 24, 0, 0, 0, 003, 0, 0)
CYLN1 <- CYLINDER (37, 24, 0, 0, 0, 003, 0, 0)
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CYLN2 <- CYLINDER (20, 18, 0, 0, 0, 050, 0, 0)
RTAP1 <- RIGHT _TAPER (30, 13, 18, 0, 0, 070, 0, 0)
UCUT2 <- UNDERCUT (05, 10, 0, 0, 0, 100, 0, 0)
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RCHM1 <- RIGHT _CHAMFER (02, 06, 08, 0, 0, 0, 148, 0, 0)
COMPONENT <- STICK - LCHM1 - CYLN1 - UCUT1 - CYLN2 - RTAP1 - UCUT2 - CYLN3$ - CYLN4 - RCHM1

! MATERIAL

SCOPE MATERIAL
SOLID FAMILY MATERIAL (LENGTH, RADIUS):MATERIAL MATERIAL - CYL (LENGTH, RADIUS) AT (ROTY = 90)
RADIUS = 25

! CYLINDER

SCOPE CYLINDER
SOLID FAMILY CYLINDER (LENGTH, RADIUS, X1, Y1, Z1, S X2, Y2, Z2):CYLINDER
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 <- CYL (LENGTH, RADIUS1) AT (ROTY = 90)
CYLINDER <- MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

! RIGHT _TAPER

SCOPE RIGHT _TAPER
SOLID FAMILY RIGHT _ TAPER (LENGTH, RADIUS1, S RADIUS2, X1, Y1, Z1, X2, Y2, Z2):RIGHT _ TAPER
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 <- CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 90)
RIGHT _ TAPER <- MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

! UNDERCUT

SCOPE UNDERCUT
SOLID FAMILY UNDERCUT (LENGTH, RADIUS, X1, Y1, Z1, S X2, Y2, Z2):UNDERCUT
OBJ1 <- CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 = CYL (LENGTH, RADIUS) AT (ROTY = 90)
UNDERCUT = MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 2S

RIGHT _ CHAMFER

SCOPE RIGHT _ CHAMFER
SOLID FAMILY RIGHT _ CHAMFER (LENGTH, RADIUS1, RADIUS2, $)
X1, Y2, Z1, X2, Y2, Z2 : RIGHT _ CHAMFER
OBJ1 = CYL (LENGTH, RADIUS) AT (ROTY = 90)
OBJ2 = CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 90)
RIGHT _ CHAMFER = MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

LEFT _ CHAMFER

SCOPE LEFT _ CHAMFER
SOLID FAMILY LEFT _ CHAMFER (LENGTH, RADIUS1, RADIUS2, $
X1, Y1, Z1, X2, Y2, Z2 : LEFT _ CHAMFER
OBJ1 = CYL (LENGTH, RADIUS) AT (ROTY = 270)
OBJ2 = CONE (LENGTH, RADIUS1, RADIUS2) AT (ROTY = 270)
LEFT _ CHAMFER = MOVE (OBJ1 - OBJ2) BY (X2, 0, 0)
RADIUS = 25

Part material details: Medium Carbon Steel Round Bar
Geometry: A round bar of dimensions L150 x R25 (mm)

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