ONE STEP COSINE-TAYLORLIKE METHOD FOR SOLVING STIFF EQUATIONS

ROKIAH @ ROZITA AHMAD¹ & NAZEERUDDIN YAacob²

Abstract. This paper discusses the derivation of an explicit Cosine-Taylorlike method for solving stiff ordinary differential equations. The formulation has resulted in the introduction of a new formula for the numerical solution of stiff ordinary differential equations. This new method needs an extra work in order to solve a number of differentiations of the function involved. However, the result produced is better than the results from the explicit classical fourth-order Runge-Kutta (RK4) and the implicit Adam-Bashforth-Moulton (ABM) methods. When compared with the previously derived Sine-Taylorlike method, the accuracy for both methods is almost equivalent.

Keywords: Explicit method; stiff ordinary differential equations; Runge-Kutta; implicit method; Adam-Bashforth-Moulton; Sine-Taylorlike

1.0 INTRODUCTION

A stiff system contains one or more fast decay processes along with relatively slow processes, such that the shortest decay “time constant” is much smaller than the total span of interest in the independent variable, which is usually “time” [1]. Systems of ordinary differential equations arise frequently in almost every discipline of science and engineering that usually exhibited stiffness characteristic, as a result of modeling and simulation activities. It has been suggested that the problem of stiffness is very

¹ Pusat Pengajian Sains Matematik, Fakulti Sains dan Teknologi, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor
² Jabatan Matematik, Fakulti Sains, Universiti Teknologi Malaysia, 81310 UTM Skudai, Johor
difficult to be solved by explicit methods but recently many explicit methods were
introduced and developed to solve the stiff problems like those in [4 - 6].

In this paper, we introduced a Taylor-like explicit method which can be used to
solve stiff problems and give a good accuracy. It was shown previously in [1] that the
explicit one-step method for stiff problems could be represented by the composition
of a polynomial and an exponential function of the form

\[ PE(t) = a_0 + t \left( a_1 + t \left( a_2 + t \left( a_3 + t (a_4 + a_5 t) \right) \right) \right) + Ah e^{\alpha t}. \]  

(1.1)

In [2], taking \( A = 1 \), we had calculated the values of \( a_i, i = 0,1,2,3,4,5 \) and \( b_i, i = 1,2 \).
In [3], we had substituted \( A \) with a trigonometric function, \( A = \sin z_n h \). Based on the
same theory for the solution of a differential equation with complex eigenvalues, we
replaced \( A \) by \( \cos z_n h \) to produce a Cosine-Taylor-like method. Provided that \( f^{(5)}, f^{(6)} \neq 0 \),
we obtain the equation

\[ \left[ PE(t) = y_n + (t - t_n) \left( f_n + (t - t_n) \left( f_n \times \left( \frac{f_n}{2} + (t - t_n) \left( \frac{f_n}{6} + (t - t_n) \left( \frac{f_n}{24} + (t - t_n) \frac{f^{(4)}}{120} \right) \right) \right) \right) \right] \]

\[ \frac{f^{(5)}}{z_n^5} \left( e^{z_n (t - t_n)} - 1 - z_n (t - t_n) - \frac{1}{2} (z_n (t - t_n))^2 - \frac{1}{6} (z_n (t - t_n))^3 - \frac{1}{24} (z_n (t - t_n))^4 \right) \]

\[ - \frac{1}{120} (z_n (t - t_n))^5 \],

(1.2)

where \( z_n = \frac{f^{(6)}}{f^{(5)}} \).

Letting \( t = t_{n+1} \), we obtain the following formula:

\[ \left[ y_{n+1} = y_n + h \left( f_n + h \left( \frac{f_n}{2} + h \left( \frac{f_n}{6} + h \left( \frac{f_n}{24} + h \frac{f^{(4)}}{120} \right) \right) \right) \right) \right] + \]

\[ \frac{f^{(5)}}{z_n^5} \cos(z_n h) \left( \exp(z_n h) - 1 - hz_n \left( \frac{1}{2} + hz_n \left( \frac{1}{6} + hz_n \left( \frac{1}{24} + \frac{h^2}{120} \right) \right) \right) \right) \].

(1.3)
2.0 STABILITY

Theorem 1.1

The explicit cosine-Taylorlike method is \( A \)-stable.

Proof:

On applying equation (1.3) to the test equation \( y' = \lambda y \) with \( \text{Re} \left( \lambda \right) < 0 \), we obtain

\[
\begin{align*}
[y_{*,i}] &= y_s + h \left( \lambda y_s + h \left( \frac{\lambda^2 y_s}{6} + h \left( \frac{\lambda^3 y_s}{24} + h \left( \frac{\lambda^4 y_s}{120} \right) \right) \right) \right) + \]

\[
\frac{\lambda^h y_s \cos(\lambda h)}{\lambda^h} \left( e^{\lambda h} - 1 - \lambda h \left( \frac{1}{2} + \lambda h \left( \frac{1}{6} + \lambda h \left( \frac{1}{24} + \lambda h \left( \frac{1}{120} \right) \right) \right) \right) \right)
\]

\[
\left[ = y_s + \lambda h y_s \left\{ 1 + \frac{\lambda h}{2} + \frac{(\lambda h)^2}{6} + \frac{(\lambda h)^3}{24} + \frac{(\lambda h)^4}{120} \right\} \right]
\]

\[
y_s \cos(\lambda h) \left( e^{\lambda h} - 1 - \lambda h \left( \frac{1}{2} + \lambda h \left( \frac{1}{6} + \lambda h \left( \frac{1}{24} + \lambda h \left( \frac{1}{120} \right) \right) \right) \right) \right)
\]

\[
\left[ = y_s \left\{ 1 + h \lambda + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{120} \right\} \right]
\]

\[
y_s \cos(\lambda h) \left( e^{\lambda h} - 1 - \lambda h \left( \frac{1}{2} + \lambda h \left( \frac{1}{6} + \lambda h \left( \frac{1}{24} + \lambda h \left( \frac{1}{120} \right) \right) \right) \right) \right)
\]

\[
\left[ = y_s \left( e^{\lambda h} \cos(\lambda h) + (1 - \cos(\lambda h)) \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!} \right) \right) \right]
\]

i.e.,

\[
y_{*,i} = y_s \left( e^{\lambda h} \cos(\lambda h) + (1 - \cos(\lambda h)) \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!} \right) \right)
\]

which gives us

\[
y_{*,i} = Q(\lambda h) y_s,
\]
where
\[ Q(\lambda h) = e^{\lambda h} \cos(\lambda h) + (1 - \cos(\lambda h)) \left( 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!} \right) \]
\[ = e^{\lambda h} \cos(\lambda h) + (1 - \cos(\lambda h)) e^{\lambda h} \]
\[ = e^{\lambda h}. \tag{2.1} \]

Since \( y_{n+1} = e^{\lambda h} y_n \) for \( n = 0,1,2,... \),
\[ y_1 = e^{\lambda h} y_0; \quad y_2 = e^{2\lambda h} y_0; \quad \ldots; \quad y_k = e^{k\lambda h} y_{k-1} = e^{k\lambda h} y_0. \]

For any fixed point \( t = t_n = nh \), we have
\[ y_n = e^{\lambda h} y_0. \]

Since \( |e^{\lambda h}| \to 0 \) as \( n \to \infty \) for all \( \lambda h \) with \( \text{Re}(\lambda) < 0 \), hence \( y_n \to 0 \) as \( n \to \infty \) and consequently the method is \( A \)-stable. \( \blacksquare \)

**Theorem 1.2**
The explicit Cosine-Taylor-like method is also \( L \)-stable.

**Proof:**
We have shown that on applying equation (1.3) to the test equation \( y' = \lambda y \), with \( \text{Re}(\lambda) < 0 \), we obtain
\[ y_{n+1} = e^{\lambda h} y_n. \]

From Theorem 1.1, the method is \( A \)-stable. Since \( |e^{\lambda h}| \to 0 \) as \( \text{Re}(\lambda h) \to -\infty \), hence it is \( L \)-stable. \( \blacksquare \)

Figure 1 and Figure 2 illustrate the stability polynomial and stability region of the method, respectively.

### 3.0 Numerical Results

The formula (1.3) was tested on the stiff ordinary differential equation
\[ y'(t) = -100y(t) + 99e^{-t}; \quad y(0) = 1, \tag{3.1} \]
and the result is compared to the exact solution which is represented by
Figure 1  The stability polynomial given by the method above

Figure 2  The stability region given by the method above
We also solved equation (3.1) using three other methods; namely the classical fourth-order Runge-Kutta (RK4), the implicit Adam-Bashforth-Moulton (ABM) and the Sine-Taylorlike (STL6)[3] methods. By taking $h = 0.02$, the relative errors for the four methods applied were compared and presented in Table 1.

The relative errors for the four methods used are plotted and illustrated in Figure 3.

4.0 DISCUSSION AND CONCLUSION

This research has generally discussed a one-step explicit method in solving stiff ordinary differential equations, namely, the Cosine-Taylorlike method. The results showed excellent relative errors of the Cosine-Taylorlike method compared to the classical Runge-Kutta and the Adam-Bashforth-Moulton methods. The one-step explicit Sine Taylorlike method, which we had derived earlier, also showed comparable relative errors to the Cosine-Taylorlike method. We have proved that the Cosine-Taylorlike method is both $A$-stable and $L$-stable. We realize that the function evaluations of the methods is about the same for every method but we believe that the cost of computation is much ‘cheaper’ in the explicit formula compared with the implicit ones, and is reflected in the efficiency of the method in dealing with stiffness. We conclude that the proposed method is comparable to our previous Sine-Taylorlike method.

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<th>Cosine-Taylorlike ($A=\sin(\pi h)$) [3]</th>
<th>Cosine-Taylorlike ($A=\cos(\pi h)$)</th>
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